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Resolutions of Determinantal Ideals: n-Minors of $(n+2)$-Square Matrices

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INTRODUCTION

Let R be a Noetherian commutative ring with unit, and x_{ij} be variables with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. If we let $S = R[x_{ij}]$ be the polynomial ring over R, then we have the generic matrix (x_{ij}) and we may form the ideal I_t of S generated by the $t \times t$ minors of this matrix for $1 \leqslant t \leqslant \min(m, n)$. Hochster and Eagon [15] proved that I_t is perfect (i.e., grade $I_t = \mathrm{pd}_S\, S/I_t$) and that S/I_t is a (normal) domain if R is a (normal) domain. So S/I_t is R-free, and if R is Cohen–Macaulay, then so is S/I_t. Therefore, free resolutions of S/I_t have the property of so-called *depth sensitivity* (see Chap. II). Svanes [26] proved that S/I_t is Gorenstein, if and only if R is Gorenstein, and $t = 1$ or $m = n$.

For many years there has been considerable interest in finding a minimal free resolution (for the definition of minimality, see Chapter II) of S/I_t. If we have a minimal free resolution $\mathbb{P}.$ of S/I_t when $R = \mathbb{Z}$, the ring of integers, then for any R, $R \otimes_{\mathbb{Z}} \mathbb{P}$ is a minimal free resolution, since S/I_t is $\mathbb{Z}$-free.

If $t = 1$, then the Koszul complex gives us such a resolution. Eagon and Northcott [9] constructed a minimal free resolution of S/I_t when $t = \min(m, n)$.

On the other hand, Roberts [23], Lascoux [17], and Pragacz and Weyman [22] constructed the minimal free resolution (Lascoux's resolution) of S/I_t for any m, n, and t in the case where R contains the rational number field $\mathbb{Q}$. Their description of the resolution is based on the representation theory of general linear group. Let F and G be free R

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modules of rank m and n, respectively. Then we may identify S with $S(F \otimes G)$, the symmetric algebra of $F \otimes G$. The group $GL(F) \times GL(G)$ acts on $S(F \otimes G)$ naturally, and I_t is invariant under the action. Each term of the resolution is expressed as a polynomial representation (for definition, see [12]) of $GL(F_S) \times GL(G_S)$, where $F_S = S \otimes_R F$ and $G_S = S \otimes_R G$.

There has been some efforts to apply the representation theory to the case $R = \mathbb{Z}$. Buchsbaum [3] gave another description of the Eagon–Northcott complex, using multilinear algebra. Akin, Buchsbaum, and Weyman [2] developed characteristic-free representation theory of general linear groups, and constructed a minimal free resolution (the Akin–Buchsbaum–Weyman complex) of S/I_t over $\mathbb{Z}$, in the case $t = \min(m, n) - 1$.

In this article, we prove that there exists a minimal free resolution of S/I_t (over $\mathbb{Z}$) in the case $m = n = t + 2$. Our main methods are also a characteristic-free representation theory of GL. Our proof consists in showing that the Betti numbers of S/I_t are independent of the characteristic of the ground field, so it does not provide an explicit construction of a resolution. The problem whether S/I_t has a minimal free resolution in the case $R = \mathbb{Z}$, $m \neq n$, and $\min(m, n) - 2 = t > 1$ is still open.

Now we describe the contents of this article.

In Chapter I, we prepare some basic facts on characteristic-free representation of GL, including Schur functors and Schur complexes. When $R = Q$, the Schur functor $L_\lambda F$ (λ a partition) of a vector space F is an irreducible polynomial representation of $GL(F)$, if $L_\lambda F$ is not zero. Schur functors are defined over any ground field, and are fundamental objects in the characteristic-free representation theory of GL. Schur complexes are generalizations of Schur functors in some sense. They are parameterized by partitions, and are sometimes useful machinery in homological methods. Although the Schur complex is defined for any finite free complex in the characteristic zero case, it is defined only for a morphism of finite free module (i.e., a finite free complex of length one) in the characteristic-free case. We define symmetric powers (Schur complexes for single-columned partitions) of a finite free complex of length two, using symmetric, exterior, and divided power algebra of finite free modules. It seems difficult to extend this functor to general Schur complexes. For Schur functors and Schur complexes in the characteristic-free case, we refer the reader to [2, 4].

In Chapter II, we introduce some basic facts on determinantal ideals: ring theoretical and homological properties. In the case $m = n = t + 2$, and $R = \mathbb{Z}$ or a field, S/I_t is Gorenstein of codimension 9. So if we denote by β_i the ith Betti number of S/I_t, it holds that $\beta_i = \beta_{9-i}$. Our main theorem (the existence of the resolution) is true, if β_i is independent of the characteristic of the ground field, for any i. On the other hand, Kurano [16] proved that β_2 is independent of the characteristic, for any m, n, and t.

So our interest in later chapters is concentrated upon the calculation of β_3 and β_4. Furthermore, if we let S be graded so that each x_{ij} is of degree one, I_t is homogeneous and the Poincaré series of S/I_t does not depend on the characteristic. This fact is important in our grade-wise calculation.

In [3], Buchsbaum defined linear complexes $\mathbb{X}^t$ and $\mathbb{Z}^t$. $\mathbb{X}^t$ and $\mathbb{Z}^t$ are linearly exact (for the definition of linear exactness, see Definition 2.6 of [1]), and if there exists a minimal free resolution of I_t (resp. I_t/I_{t+1}), then $\mathbb{X}^t$ (resp. $\mathbb{Z}^t$) is isomorphic to the linear part of the minimal free resolution of I_t (resp. I_t/I_{t+1}). His method is generalized in [10]. In [1], $\mathbb{X}^t$ and $\mathbb{Z}^t$ are defined in a different way, and they are canonically isomorphic to the ones defined in [3] respectively, in the submaximal case (i.e., in the case $\min(m, n) = t+1$). Though $\mathbb{X}^t$ and $\mathbb{Z}^t$ defined in [3] are free and have the same rank as those defined in [1], in the case $R = \mathbb{Z}$ and $m = n = t+2$, the canonical morphism defined in Remark 3.19 of [1] does not give the isomorphisms between them. So we use the definition of [3].

In Chapter III, we extend the Cauchy formulas to the version of chain complexes. This formula plays a crucial role in Chapter IV in studying the lower syzygies of determinantal ideals. Cauchy formulas are the characteristic-free decompositions $S(F \otimes G) \simeq \sum_\lambda L_\lambda F \otimes L_\lambda G$ and $\Lambda(F \otimes G) \simeq \sum_\lambda L_\lambda F \otimes K_\lambda G$ which hold up to filtrations. Characteristic-free decomposition of $S(F \otimes G)$ was first proved by Doubilet, Rota, and Stein [8] and formulated with Schur functors by Akin, Buchsbaum, and Weyman [2]. The decomposition of $\Lambda(F \otimes G)$ first appeared in [2]. We prove the decomposition $S(\varphi \otimes \psi) \simeq \sum_\lambda L_\lambda \varphi \otimes L_\lambda \psi$ which holds up to filtrations, where φ and ψ are morphisms of finite free R-modules, and $S(\varphi \otimes \psi)$ is the symmetric algebra of $\varphi \otimes \psi$ which is defined in Chapter I. Our proof is quite analogous to the proof in [2], in which Hopf algebra structures play important roles. In fact, pairings ϕ^S and ψ^Λ agree with the pairings which appear in [2], up to sign. We use the pairing $\theta\colon \Lambda\varphi \otimes \Lambda\psi \to S(\varphi \otimes \psi)$ in place of $\phi^S\colon \Lambda F \otimes \Lambda G \to S(F \otimes G)$. θ has a similar property to the one ϕ^S has, which is formulated as the definition of Cauchy pairing.

In Chapter IV, we calculate the third Betti number of S/I_t, given some conditions on m, n, and t. We also try some gradewise calculation of the fourth Betti number of S/I_t. In [16], Kurano's method is based on the Cauchy formula for $S(F \otimes G)$. The idea of our proof comes from the proof in [16], using the natural filtration with the Cauchy formula, while our proof is based on the Cauchy formula of the complexes proved in Chap. III.

In Chapter V, we prove that the Betti numbers of S/I_t are independent of the characteristic if $m = n = t+2$. Now the problem is the calculation of β_4. Using the arguments in Chap. II and the calculations in Chapter IV, we can prove the case $t \geqslant 5$ without much difficulties. But we need more argument in proving the case $2 \leqslant t \leqslant 4$. We first prove that the rank of

X_5^t is independent of the characteristic in our case. This fact enables us to complete the proof of the case $2 \leqslant t \leqslant 3$. To complete the proof in the case $t=4$, we analyze the resolution of the case $t=5$, and reduce to this case.

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I. PRELIMINARIES

Partitions and Tableaux

We denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of positive integers (resp. nonnegative integers). And we denote by $\mathbb{Z}$ (resp. $\mathbb{Q}$) the ring of integers (resp. the field of rational numbers). For a set X, $\# X$ will stand for the cardinality of X. For $k \in \mathbb{N}$, we will denote by $\mathfrak{S}_k$ the symmetric group on $\{1, ..., k\}$.

We set $\Omega^* = \mathrm{Hom}_{\mathrm{set}}(\mathbb{N}, \mathbb{Z})$. So if $\lambda \in \Omega^*$, λ is a sequence of integers: $\lambda = (\lambda_1, \lambda_2, ...)$. Let λ and μ be elements of Ω^* and $k \in \mathbb{Z}$. We define $\lambda + \mu$ to be the sequence $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, ...)$ and $k \cdot \lambda$ to be the sequence $(k \cdot \lambda_1, k \cdot \lambda_2, ...)$. We also define $\mathrm{supp}\, \lambda = \{i \in \mathbb{N} \mid \lambda_i \neq 0\}$. We denote by Ω^+ the set $\{\lambda \in \Omega^* \mid {}^\forall i \in \mathbb{N}\ \lambda_i \in \mathbb{N}_0 \text{ and } \#(\mathrm{supp}\, \lambda) < \infty\}$. If λ is an element of Ω^+, we put $\mathrm{lg}(\lambda) = \max \mathrm{supp}\, \lambda$ and $|\lambda| = \sum_{i \in \mathbb{N}} \lambda_i$ (if $\lambda = (0, 0, ...)$, then we define $\mathrm{lg}(\lambda) = 0$). We call $\mathrm{lg}(\lambda)$ *length* of λ and $|\lambda|$ *weight* of λ. Let $k \in \mathbb{N}_0$. We put $\Omega_k^+ = \{\lambda \in \Omega^+ \mid |\lambda| = k\}$, $\Omega^- = \{\lambda \in \Omega^+ \mid {}^\forall i \in \mathbb{N}\ \lambda_i \geqslant \lambda_{i-1}\}$, and $\Omega_k^- = \Omega_k^+ \cap \Omega^-$. We call an element of Ω^- a *partition*. It is easy to check that Ω_k^- is a finite set for any k.

DEFINITION I.1.1. For any $\lambda \in \Omega^+$, $\tilde{\lambda}$ is defined to be the partition which satisfies $\tilde{\lambda}_i = \#\{j \in \mathbb{N} \mid \lambda_j \geqslant i\}$ for any $i \in \mathbb{N}$. $\tilde{\lambda}$ is called the *transpose* of λ.

We now define some elements of Ω^* which play special roles later. We denote by 0 the element $(0, 0, ...) \in \Omega_0^-$. Let $k \in \mathbb{N}$. We define the element ε_k of Ω_1^+ given by ${}^\forall i \in \mathbb{N}\ (\varepsilon_k)_i = \delta_{ki}$ (Kronecker's δ).

We now set $\omega_k = \sum_{i=1}^k \varepsilon_i$ and $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$. It holds that $\omega_k = (k \cdot \varepsilon_1)^\sim$.

For $\lambda, \mu \in \Omega^+$, we say that $\lambda \supset \mu$ if and only if ${}^\forall i \in \mathbb{N}\ \lambda_i \geqslant \mu_i$. It is clear that $\supset$ is an order and $\lambda \supset \mu$ implies $\lambda \geqslant \mu$ (see below). If λ, $\mu \in \Omega^-$, then it holds that $\lambda \supset \mu$ if and only if $\tilde{\lambda} \supset \tilde{\mu}$.

Let λ, $\mu \in \Omega^-$ with $\lambda \supset \mu$. We define the subset $S_\square(\lambda/\mu)$ of Ω^+ by

$$S_\square(\lambda/\mu) = \{\nu \in \Omega^+ \mid {}^\exists t \in \mathbb{N}, {}^\exists k \in \mathbb{N}_0 \quad k < \lambda_{t+1} - \mu_t \quad \text{and} \\ \nu = \lambda - \mu + (\lambda_{t+1} - \mu_{t+1} - k) \cdot \alpha_t\}.$$

$S_\square(\lambda)$ stands for $S_\square(\lambda/0)$.

LEMMA I.1.2. *If* $k \in \mathbb{N}_0$ *and* $\lambda \in \Omega_k^+$ *then*

(1) $|\tilde{\lambda}| = k$

(2) *there exists* $\sigma \in \mathfrak{S}_{\lg(\lambda)}$ *such that* $\tilde{\tilde{\lambda}}_i = \lambda_{\sigma(i)}$ $(i \in \mathbb{N})$

(3) $\tilde{\tilde{\lambda}} = \lambda$ *if and only if* λ *is a partition.*

Proof. Easy.

We introduce the lexicographic order into Ω^+. Namely, if λ and μ are elements of Ω^+, then we say that $\lambda > \mu$ when $^\exists i \in \mathbb{N}$ $^\forall j < i$ $\lambda_j = \mu_j$ and $\lambda_i > \mu_i$. We say that $\lambda \geqslant \mu$ when $\lambda > \mu$ or $\lambda = \mu$. With this ordering $\geqslant$, Ω^+ is a totally ordered set. Note that if $\lambda \in \Omega^-$ and $\mu \in S_\square(\lambda)$, then $\tilde{\tilde{\mu}} > \lambda$.

DEFINITION I.1.3. The *diagram* (or *shape*) of an element $\lambda \in \Omega^+$ is the set $\{(i, j) \in \mathbb{N}^2 \mid j \leqslant \lambda_i\}$, and is denoted by Δ_λ. Here we use the convention that is used with matrices, namely, that the row index i increases as one goes downward, and the column index j increases from left to right. The *skew-shape* of a pair λ, $\mu \in \Omega^+$, such that $\lambda \supset \mu$, is $\Delta_\lambda - \Delta_\mu$ and is denoted by $\Delta_{\lambda/\mu}$.

Clearly, $\Delta_{\lambda/0} = \Delta_\lambda$. So we will always adapt the terminology of skew-shape to the terminology of shape by letting $\mu = 0$.

DEFINITION I.1.4. Let X be a totally ordered set and let $\lambda, \mu \in \Omega^+$. We define $\mathrm{Tab}_{\lambda/\mu}(X)$ to be the set $\mathrm{Hom}_{\mathrm{set}}(\Delta_{\lambda/\mu}, X)$. An element of $\mathrm{Tab}_{\lambda/\mu}(X)$ is called a *tableau of shape* λ/μ *with values in the set* X.

For a tableau $T \in \mathrm{Tab}_{\lambda/\mu}(X)$ and subsets $I \subset X$ and $N \subset \mathbb{N}$, we denote $\#\{(i, j) \in \Delta_{\lambda/\mu} \mid i \in N \text{ and } T(i, j) \in I\}$ by $n_N(T, I)$. In this notation, an element $x \in X$ (resp. $i \in \mathbb{N}$) may stand for the singleton $\{x\}$ (resp. $\{i\}$).

2. *Hopf Algebras in the Category* $\mathscr{C}$

Let R be a commutative ring with unit. If there will be no confusion, $\otimes_R$, tensor product over R, will be denoted simply by $\otimes$. We always assume that R is $\mathbb{Z}$-graded (resp. $\mathbb{Z}^2$-graded) in a trivial way; i.e., we assume that the degree 0 (resp. $(0, 0)$) part of R is R itself, and that the other homogeneous parts are 0. We denote by G_R (resp. G_R^2) the category of $\mathbb{Z}$-graded (resp. $\mathbb{Z}^2$-graded) R-modules. We let $\mathscr{C}$ stand for the category of chain complexes in the abelian category G_R. The class of objects of a category $\mathscr{A}$ may again be denoted by $\mathscr{A}$ if there will be no confusion. Let $A \in \mathscr{C}$. We can express A as

$$A = \cdots \to A_{j+1} \xrightarrow{\partial_{j+1}^A} A_j \xrightarrow{\partial_j^A} A_{j-1} \to \cdots.$$

A_j is the degree j part of A, and ∂_j^A is the boundary map of A for each degree j. We denote by ∂^A the boundary map $\sum_j \partial_j^A : A \to A$. Since $A_j \in G_R$ for each j, one can decompose A_j into the direct sum $A_j = \sum_{i \in \mathbb{Z}} A_j^i$. We call A_j^i the degree (i, j) part of A (i is the degree as an graded R-module, and j is the degree as an R-complex). So one can assume A is an object of G_R^2 (forgetting the boundary map). From this viewpoint, an object A in $\mathscr{C}$ is an object in G_R^2 equipped with a morphism $\partial^A : A \to A$ of degree $(0, -1)$ such that $\partial^A \circ \partial^A = 0$. We can convert $\otimes$ into a functor $G_R^2 \times G_R^2 \to G_R^2$. Let $A = \sum_{i,j} A_j^i$ and $B = \sum_{i,j} B_j^i$ be objects of G_R^2. We define the grading of $A \otimes B$ by the total grading. Namely, we define $(A \otimes B)_j^i = \sum_{i_1 + i_2 = i, j_1 + j_2 = j} A_{j_1}^{i_1} \otimes B_{j_2}^{i_2}$. Furthermore, if A and B are objects of $\mathscr{C}$, we define $\partial^{A \otimes B}$ given by $\partial^{A \otimes B}(x \otimes y) = \partial^A(x) \otimes y + (-1)^{j_1} \cdot x \otimes \partial^B(y)$, for $x \in A_{j_1}^{i_1}$ and $y \in B_{j_2}^{i_2}$. It is easy to see that $A \otimes B$ is converted into an object of $\mathscr{C}$ with this boundary, and $\otimes$ is functorial in A and B.

Let $A, B \in G_R^2$. We have a *twisting morphism* $T_{A,B} : A \otimes B \to B \otimes A$ given by $T_{A,B}(x \otimes y) = (-1)^{i_1 i_2 + j_1 j_2} y \otimes x$ for $x \in A_{j_1}^{i_1}$ and $y \in B_{j_2}^{i_2}$. It is easy to check that

LEMMA I.2.1. *T is natural in A and B. If $A, B \in \mathscr{C}$, then $T_{A,B}$ is a chain map. We have the identity $T_{B,A} \circ T_{A,B} = \mathrm{id}_{A \otimes B}$. If C is also an object in G_R^2, then the following diagram is commutative:*

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, "T_{A \otimes B, C}"] \arrow[d, "\simeq"] & & C \otimes (A \otimes B) \arrow[dd, "\simeq"] \\
A \otimes (B \otimes C) \arrow[d, "\mathrm{id}_A \otimes T_{B,C}"] & & \\
A \otimes (C \otimes B) \arrow[r, "\simeq"] & (A \otimes C) \otimes B \arrow[r, "T_{A,C} \otimes \mathrm{id}_B"] & (C \otimes A) \otimes B
\end{tikzcd}$$

We can make G_R^2 and $\mathscr{C}$ into symmetric categories with $\otimes$ and T (for the definition of symmetric category, see [19]). Now we can define *algebra*, *coalgebra*, and *Hopf algebra* (or *bialgebra*) in the categories G_R^2 and $\mathscr{C}$ as in [4]. For a Hopf algebra $A = \sum_i A^i$ (in G_R or G_R^2 or $\mathscr{C}$), we always require that $A^0 = R$, and that the unit morphism (resp. co-unit morphism) be the natural inclusion map $R = A^0 \subsetneq A$ (resp. the natural projection $A \to A^0 = R$). We denote by m_A (resp. Δ_A) the multiplication (resp. comultiplication) of A. If there is no danger of confusion, it will be simply denoted by m (resp. Δ). For a Hopf algebra in G_R (resp. G_R^2, $\mathscr{C}$) we always require that m_A and Δ_A be morphisms in the category G_R (resp. G_R^2, $\mathscr{C}$).

We can also define a *tensor product* of two (co-, Hopf) algebras and *commutativity* of (co-, Hopf) algebra via the twisting morphism T as in [4].

Let F be a finite free R-module. We denote by $SF = \sum_i S_i F$ (resp. $\Lambda F = \sum_i \Lambda^i F$, $DF = \sum_i D_i F$) the symmetric (resp. exterior, divided power) algebra, where S_iF (resp. Λ^iF, D_iF) is the ith symmetric (resp. exterior, divided) power of F. We always assume that $S_iF = \Lambda^iF = D_iF = 0$ if $i < 0$. SF, ΛF, and DF have the structures of Hopf algebras. The bialgebra DF is defined to be the graded dual (cf. [21]) of SF^*, where F^* is $\mathrm{Hom}_R(F, R)$. The comultiplications of these three Hopf algebras are homomorphisms of algebras induced by the diagonal map given by $f \in F \mapsto f \oplus f \in F \oplus F$ (note that $S(F \oplus F) \simeq SF \otimes SF$, $\Lambda(F \oplus F) \simeq \Lambda G \otimes \Lambda G$, and $D(F \oplus F) \simeq DF \otimes DF$). So they are sometimes called *diagonalizations*. S, Λ, and D are functors from the category of finite free R-modules to the category of Hopf algebras.

Now let $\varphi: G \to F$ be a morphism of finite free R-modules. We associate $S\varphi$ and $\Lambda\varphi$ with the morphism φ. $S\varphi = SF \otimes \Lambda G$ and $\Lambda\varphi = \Lambda F \otimes DG$ are Hopf algebras in the category G_R^2, where we assume that S_iF, Λ^jG, Λ^iF, and D_jG are of degree $(2i, 0)$, $(2j, j)$, $(i, 0)$, and (j, j), respectively, so that SF, ΛG, ΛF, and DG are Hopf algebras in the category G_R^2. With this grading, $S\varphi$ and $\Lambda\varphi$ are commutative. Note that the structures of $S\varphi$ and $\Lambda\varphi$ as Hopf algebras are not dependent on the morphism φ. They depends only on the modules F and G. It is clear that S and Λ are functors from the category of morphisms of finite free R-modules to the categories of Hopf algebras in G_R^2. $S\varphi$ and $\Lambda\varphi$ are converted into Hopf algebras in the category $\mathscr{C}$ as follows. The boundary map of S_φ is the composition of the maps

$$\partial^{S\varphi}: S\varphi = SF \otimes \Lambda G \xrightarrow{\mathrm{id}_{SF} \otimes \Delta_{\Lambda G}} SF \otimes \Lambda^1 G \otimes \Lambda G$$

$$\xrightarrow{\mathrm{id}_{SF} \otimes \varphi \otimes \mathrm{id}_{\Lambda G}} SF \otimes S_1 F \otimes \Lambda G \xrightarrow{m_{SF} \otimes \mathrm{id}_{\Lambda G}} SF \otimes \Lambda G = S\varphi,$$

where we identify $\Lambda^1 G = G$ and $S_1 F = F$. Similarly, the boundary map of $\Lambda\varphi$ is the composition of the maps

$$\partial^{\Lambda\varphi}: \Lambda\varphi = \Lambda F \otimes DG \xrightarrow{\mathrm{id}_{\Lambda F} \otimes \Delta_{DG}} \Lambda F \otimes D_1 G \otimes DG$$

$$\xrightarrow{\mathrm{id}_{\Lambda F} \otimes \varphi \otimes \mathrm{id}_{DG}} \Lambda F \otimes \Lambda^1 F \otimes DG \xrightarrow{m_{\Lambda F} \otimes \mathrm{id}_{DG}} \Lambda F \otimes DG = \Lambda\varphi.$$

LEMMA I.2.2. *$\partial^{S\varphi} \circ \partial^{S\varphi} = 0$ and $\partial^{\Lambda\varphi} \circ \partial^{\Lambda\varphi} = 0$. With these boundaries, $m_{S\varphi}$, $\Delta_{S\varphi}$, $m_{\Lambda\varphi}$, and $\Delta_{\Lambda\varphi}$ are chain maps so that $S\varphi$ and $\Lambda\varphi$ are converted into Hopf algebras in the category $\mathscr{C}$. Moreover, S and Λ are converted into functors from the category of morphisms of finite free R-module to the category of Hopf algebras in $\mathscr{C}$. $m_{S\varphi}$, $\Delta_{S\varphi}$, $m_{\Lambda\varphi}$, and $\Delta_{\Lambda\varphi}$ are converted into natural transformations.*

Proof. Easy.

For $i \in \mathbb{N}_0$, we denote by $S_i\varphi$ the subcomplex of $S\varphi$ given by

$$0 \to \Lambda^i G \to F \otimes \Lambda^{i-1} G \to \cdots \to S_{i-1} F \otimes G \to S_i F \to 0.$$

$S\varphi$ is the graded R-complex $\sum_{i \in \mathbb{N}_0} S_i\varphi$. $S_i\varphi$ is the degree $2i$ component of $S\varphi$. Similarly, we denote by $\Lambda^i\varphi$ the subcomplex of $\Lambda\varphi$ given by

$$0 \to D_i G \to F \otimes D_{i-1} G \to \cdots \to \Lambda^{i-1} F \otimes G \to \Lambda^i F \to 0.$$

$\Lambda^i\varphi$ is the degree i component of $\Lambda\varphi$. Note that $\Lambda^i\varphi$ is isomorphic to $(S_i\varphi^*)^*$ as an R-complex. We always assume that $S_i\varphi = \Lambda^i\varphi = 0$ if $i < 0$.

The rest of this section is devoted to extending the definition of the functor S. Let α: $0 \to G \xrightarrow{\psi} F \xrightarrow{\varphi} E \to 0$ be a chain complex of finite free R-modules. We define $S\alpha$ as the Hopf algebra in the category G_R^2 given by $S\alpha = SE \otimes \Lambda F \otimes DG$, where we assume that $S_i E$, $\Lambda^j F$, and $D_k G$ are of degree $(2i, 0)$, $(2j, j)$, and $(2k, 2k)$, respectively. With this grading, $S\alpha$ is commutative. $S\alpha$ is also converted into a chain complex. The boundary is given by

$$\partial_1^{S\alpha}: (S\alpha)_1 = \sum_{j+2k=1} SE \otimes \Lambda^j F \otimes D_k G$$

$$\xrightarrow{\partial^{S\varphi} \otimes id_{DG} + (-1)^j id_{SE} \otimes \partial^{\Lambda\psi}} \sum_{j+2k=l-1} SE \times \Lambda^j F \otimes D_k G = (S\alpha)_{l-1}.$$

It is easy to check that $\partial_{l-1}^{S\alpha} \circ \partial_l^{S\alpha} = 0$. Moreover,

LEMMA I.2.3. *With the boundary $\partial^{S\varphi}$ defined above, $m_{S\alpha}$ and $\Delta_{S\alpha}$ are chain maps. S is a functor from the category of finite free R-complex (= bounded R-complex with each term being a finite free R-module) with all terms being 0 except for degree 0, 1, and 2 to the category of Hopf algebra in $\mathscr{C}$. $m_{S\alpha}$ and $\Delta_{S\alpha}$ are natural transformations.*

Proof. At first, we will show that $\partial^{S\varphi} \circ m_{S\alpha} = m_{S\alpha} \circ \partial^{S\varphi \otimes S\varphi}$. To show this, it is sufficient to check that each component

$$SE \otimes \Lambda^{j_1} F \otimes D_{k_1} G \otimes SE \otimes \Lambda^{j_2} F \otimes D_{k_2} G \to SE \otimes \Lambda^{j_1+j_2-1} F \otimes D_{k_1+k_2} G$$

and

$$SE \otimes \Lambda^{j_1} F \otimes D_{k_1} G \otimes SE \otimes \Lambda^{j_2} F \otimes D_{k_2} G \to SE \otimes \Lambda^{j_1+j_2+1} F \otimes D_{k_1+k_2-1} G$$

of both sides coincide for $j_1, j_2, k_1, k_2 \in \mathbb{N}_0$. The coincidence for the first component follows from the fact that $m_{S\varphi}$ is a chain map. The left hand side for the second component is the composition of the maps

$$
\begin{aligned}
&SE\otimes \Lambda^{j_1}F\otimes D_{k_1}G\otimes SE\otimes \Lambda^{j_2}F\otimes D_{k_2}G\\
&\qquad\xrightarrow{T} SE\otimes SE\otimes \Lambda^{j_1}F\otimes D_{k_1}G\otimes \Lambda^{j_2}F\otimes D_{k_2}G\\
&\qquad\xrightarrow{(-1)^{k_1j_2}m_{SE}\otimes m_{\Lambda\psi}} SE\otimes \Lambda^{j_1+j_2}F\otimes D_{k_1+k_2}G\\
&\qquad\xrightarrow{(-1)^{j_1+j_2}\mathrm{id}\otimes \partial^{\Lambda\psi}} SE\otimes \Lambda^{j_1+j_2+1}F\otimes D_{k_1+k_2-1}G,
\end{aligned}
$$

where T is an appropriate twisting. The sign $(-1)^{k_1j_2}$ comes out from the difference between the grading of ΛF and DG in the definition of $\Lambda\psi$ and the grading in the definition of $S\alpha$. The right hand side of the second component is the composition of the maps

$$
\begin{array}{c}
SE\otimes \Lambda^{j_1}F\otimes D_{k_1}G\otimes SE\otimes \Lambda^{j_2}F\otimes D_{k_2}G\\
\Big\downarrow {\scriptstyle (-1)^{j_1}\mathrm{id}_{SE}\otimes\partial_{\Lambda\psi}\otimes\mathrm{id}+(-1)^{j_1+j_2}\mathrm{id}\otimes\mathrm{id}_{SE}\otimes\partial_{\Lambda\psi}}\\
(SE\otimes \Lambda^{j_1+1}F\otimes D_{k_1-1}G\otimes SE\otimes \Lambda^{j_2}F\otimes D_{k_2}G)\oplus(SE\otimes \Lambda^{j_1}F\otimes D_{k_1}G\otimes SE\otimes \Lambda^{j_2+1}F\otimes D_{k_2-1}G)\\
\Big\downarrow {\scriptstyle T}\\
SE\otimes SE\otimes(\Lambda^{j_1+1}F\otimes D_{k_1-1}G\otimes \Lambda^{j_2}F\otimes D_{k_2}G\otimes \Lambda^{j_1}F\otimes D_{k_1}G\otimes \Lambda^{j_2+1}F\otimes D_{k_2-1}G)\\
\Big\downarrow {\scriptstyle m_{SE}\otimes((-1)^{(k_1-1)j_2}m_{\Lambda\psi}+(-1)^{k_1(j_2+1)}m_{\Lambda\psi})}\\
SE\otimes \Lambda^{j_1+j_2+1}F\otimes D_{k_1+k_2-1}G.
\end{array}
$$

Using the fact that $m_{\Lambda\psi}$ is a chain map, one can show easily that the two maps coincide. The proof for $\Delta_{S\alpha}$ is quite similar. The second assertion is now trivial. Since $S\alpha$ is a commutative Hopf algebra, $m_{S\alpha}$ and $\Delta_{S\alpha}$ are homomorphisms of Hopf algebra. Naturalities follow from the fact that m_{SE}, $m_{\Lambda F}$, m_{DG}, Δ_{SE}, $\Delta_{\Lambda F}$, Δ_{DG}, and T are all natural. Q.E.D.

We denote by $S_i\alpha$ the degree $2\cdot i$ part of $S\alpha$, for $i\in\mathbb{Z}$. It is clear that $S\alpha=\sum_{i\in\mathbb{N}_0}S_i\alpha$.

3. *Schur Functors and Schur Complexes*

DEFINITION I.3.1. If F is a finite free R-module, and $\lambda,\mu\in\Omega^+$ with $\lg(\lambda-\mu)\leqslant q$, then we define $S_{\lambda/\mu}F$, $\Lambda_{\lambda/\mu}F$, $D_{\lambda/\mu}F$ as follows:

$$
\begin{aligned}
S_{\lambda/\mu}F&=S_{\lambda_1-\mu_1}F\otimes\cdots\otimes S_{\lambda_q-\mu_q}F\\
\Lambda_{\lambda/\mu}F&=\Lambda^{\lambda_1-\mu_1}F\otimes\cdots\otimes \Lambda^{\lambda_q-\mu_q}F\\
D_{\lambda/\mu}F&=D_{\lambda_1-\mu_1}F\otimes\cdots\otimes D_{\lambda_q-\mu_q}F.
\end{aligned}
$$

Since $S_0F = \Lambda^0F = D_0F = R$, this definition does not depend on the choice of q. If $\lambda \not\supset \mu$, then $S_{\lambda/\mu}F = \Lambda_{\lambda/\mu}F = D_{\lambda/\mu}F = 0$ by definition.

For λ, $\mu \in \Omega^-$, let $s = \tilde{\lambda}_1$ and $t = \lambda_1$ and let (a_{ij}) be the set $s \times t$ matrix given by $a_{ij} = 1$ if $(i, j) \in \Delta_{\lambda/\mu}$ and $a_{ij} = 0$ otherwise. We denote by $d_{\lambda/\mu}(F)$ the composition of maps

$$(*) \qquad \Lambda_{\lambda/\mu}F = \Lambda_{\lambda_1 - \mu_1}F \otimes \cdots \otimes \Lambda_{\lambda_s - \mu_s}F$$

$$\xrightarrow{\Delta \otimes \cdots \otimes \Delta} (\Lambda^{a_{11}}F \otimes \cdots \otimes \Lambda^{a_{1t}}F) \otimes \cdots \otimes (\Lambda^{a_{s1}}F \otimes \cdots \otimes \Lambda^{a_{st}}F)$$

$$\longrightarrow (S_{a_{11}}F \otimes \cdots \otimes S_{a_{1t}}F) \otimes \cdots \otimes (S_{a_{s1}}F \otimes \cdots \otimes S_{a_{st}}F)$$

$$\longrightarrow (S_{a_{11}}F \otimes \cdots \otimes S_{a_{s1}}F) \otimes \cdots \otimes (S_{a_{1t}}F \otimes \cdots \otimes S_{a_{st}}F)$$

$$\xrightarrow{m \otimes \cdots \otimes m} S_{\tilde{\lambda}_1 - \tilde{\mu}_1}F \otimes \cdots \otimes S_{\tilde{\lambda}_t - \tilde{\mu}_t}F = S_{\lambda/\mu}F,$$

where the second map is induced by identification $\Lambda^{a_{ij}}F = S_{a_{ij}}F$ and the third map is the permutation according to the index a_{ij}. We denote by $d'_{\lambda/\mu}(F)$ the similar composite map

$$D_{\lambda/\mu}F = D_{\lambda_1 - \mu_1}F \otimes \cdots \otimes D_{\lambda_s - \mu_s}F$$

$$\xrightarrow{\Delta \otimes \cdots \otimes \Delta} (D_{a_{11}}F \otimes \cdots \otimes D_{a_{1t}}F) \otimes \cdots \otimes (D_{a_{s1}}F \otimes \cdots \otimes D_{a_{st}}F)$$

$$\longrightarrow (\Lambda^{a_{11}}F \otimes \cdots \otimes \Lambda^{a_{1t}}F) \otimes \cdots \otimes (\Lambda^{a_{s1}}F \otimes \cdots \otimes \Lambda^{a_{st}}F)$$

$$\longrightarrow (\Lambda^{a_{11}}F \otimes \cdots \otimes \Lambda^{a_{s1}}F) \otimes \cdots \otimes (\Lambda^{a_{1t}}F \otimes \cdots \otimes \Lambda^{a_{st}}F)$$

$$\xrightarrow{m \otimes \cdots \otimes m} \Lambda_{\tilde{\lambda}_1 - \tilde{\mu}_1}F \otimes \cdots \otimes \Lambda_{\tilde{\lambda}_t - \tilde{\mu}_t}F = \Lambda_{\lambda/\mu}F.$$

DEFINITION I.3.2. (Schur functors and coSchur functors). $\operatorname{Im}(d_{\lambda/\mu}(F))$ (resp. $\operatorname{Im}(d'_{\lambda/\mu}(F))$) is denoted by $L_{\lambda/\mu}F$ (resp. $K_{\lambda/\mu}F$). $L_{\lambda/\mu}$ (resp. $K_{\lambda/\mu}$) is called *Schur* (resp. *coSchur*) *functor with respect to the skew shape* λ/μ.

Note that the identifications $\Lambda^{a_{ij}}F = S_{a_{ij}}F$ and $D_{a_{ij}}F = \Lambda^{a_{ij}}F$ are inhomogeneous. So we can define the degrees neither of the Schur functors nor of the coSchur functors in a trivial way. But Theorem I.3.11 will enable us to define the degrees of the Schur functors and the coSchur functors. In fact, the theorem will show us that the Schur functors and the coSchur functors are naturally isomorphic to the cokernel of the natural transformations between the functors to the category G_R or G_R^2.

If $R = \mathbb{Q}$, then $K_{\lambda/\mu}F$ is isomorphic to $L_{\tilde{\lambda}/\tilde{\mu}}F$ as a $GL(F)$-module, and is irreducible if $\mu = 0$.

The definition of Schur complex is quite similar to that of Schur functor. Let $\varphi: G \to F$ be a morphism of a finite free R-module, and λ, $\mu \in \Omega^+$ with $\lg(\lambda - \mu) \leqslant q$.

DEFINITION I.3.3. We define $S_{\lambda/\mu}\varphi$ and $\Lambda_{\lambda/\mu}\varphi$ as follows:

$$S_{\lambda/\mu}\varphi = S_{\lambda_1-\mu_1}\varphi \otimes \cdots \otimes S_{\lambda_q-\mu_q}\varphi$$
$$\Lambda_{\lambda/\mu}\varphi = \Lambda^{\lambda_1-\mu_1}\varphi \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}\varphi.$$

Now let $\lambda, \mu \in \Omega^-$. We define $d_{\lambda/\mu}(\varphi)$ as the similar composite map to $d_{\lambda/\mu}(F)$. We have only to replace every F by φ in the definition (*). The *Schur complex of* φ *with respect to the skew shape* λ/μ is $\operatorname{Im}(d_{\lambda/\mu}\varphi)$. If $G=0$, then $L_{\lambda/\mu}\varphi = L_{\lambda/\mu}F$. If $F=0$, then $L_{\lambda/\mu}\varphi = K_{\lambda/\mu}G$ in degree $|\lambda-\mu|$. Now let $\operatorname{rank} F=m$ and $\operatorname{rank} G=n$. Let us fix an ordered basis $X=\{x_1<\cdots<x_m\}$ of F and an ordered basis $Y=\{y_1<\cdots<y_n\}$ of G. We put $Z=X\cup Y$ and let Z be a totally ordered set for which $X<Y$, that is, for which $x_i<y_j$ for any i and j. For $\lambda, \mu\in\Omega^+$ with $\lambda\supset\mu$ and $S\in \mathrm{Tab}_{\lambda/\mu}(X)$, we obtain an element $X_S=x_{S_1}\otimes\cdots\otimes x_{S_q}\in\Lambda_{\lambda/\mu}F$, where $q=\lg(\lambda-\mu)$ and $x_{S_i}=S(i,\mu_i+1)\wedge\cdots\wedge S(i,\lambda_i)$. For $T\in\mathrm{Tab}_{\lambda/\mu}(Y)$, we obtain an element $Y_T=y_{T_1}\otimes\cdots\otimes y_{T_q}\in D_{\lambda/\mu}G$, where $y_{T_i}=y_1^{(t^i_1)}\cdot\ \cdots\ \cdot y^{(t^i_n)}$, where $t^i_j=n_i(T,y_j)$.

DEFINITION I.3.4. A tableau $S\in\mathrm{Tab}_{\lambda/\mu}(X)$ is called *row-standard* if the rows of S are strictly increasing, i.e., if for all $i=1,\dots,q$ we have $S(i,\mu_i+1)<S(i,\mu_i+2)<\cdots<S(i,\lambda_i)$. The tableau S is called *column-standard* if the colums of S are non-decreasing, i.e., if for all $j=1,\dots,t$ $(t=\lambda_1)$ we have $S(i,j)\leqslant S(i+1,j)$ whenever (i,j) and $(i+1,j)$ are both in $\Delta_{\lambda/\mu}$. S is called *standard* if it is both row- and column-standard.

DEFINITION I.3.5. A tableau $T\in\mathrm{Tab}_{\lambda/\mu}(Y)$ is called *co-row-standard* if the rows of T are non-decreasing, and *co-column-standard* if the columns of T are strictly increasing. T is called *co-standard* if it is co-row- and co-column-standard.

Clearly, the set $\{X_S\mid S\in\mathrm{Tab}_{\lambda/\mu}(X)$ and S is row-standard$\}$ is a free basis of $\Lambda_{\lambda/\mu}F$ and the set $\{Y_T\mid T\in\mathrm{Tab}_{\lambda/\mu}(Y)$ and T is co-row-standard$\}$ is a free basis of $D_{\lambda/\mu}G$.

For $U\in\mathrm{Tab}_{\lambda/\mu}(Z)$, we obtain an element $Z_U=z_{U_1}\otimes\cdots\otimes z_{U_q}$, with $z_{U_i}=\varepsilon_{U_i}\cdot U(i,\alpha_1)\wedge\cdots\wedge U(i,\alpha_1)\otimes y_1^{(t^i_1)}\cdot\ \cdots\ \cdot y_n^{(t^i_n)}$, where $\varepsilon_{U_i}=(-1)^{\#\{(\alpha,\beta)\mid\alpha>\beta,\, U(i,\alpha)\in X,\, U(i,\beta)\in Y\}}$, $\{\alpha_1,\dots,\alpha_1\}=\{j\mid(i,j)\in\Delta_{\lambda/\mu}$ and $U(i,j)\in X\}$ with $\alpha_1\leqslant\cdots\leqslant\alpha_l$, and $t^i_j=n_i(U,y_j)$ for each j.

DEFINITION I.3.6. A tableau $U\in\mathrm{Tab}_{\lambda/\mu}(Z)$ is called *row-standard* mod Y if each row of U is non-decreasing, and if, when repeats occur in a row, they occur only among elements of Y. U is *column-standard* mod Y if each column is non-decreasing, and if, when repeats occur in a column, they occur only among elements in $Z - Y = X$. U is *standard* mod Y if T is row- and column-standard mod Y.

We let $\mathrm{Row}_{\lambda/\mu}(Z, Y) = \{U \in \mathrm{Tab}_{\lambda/\mu}(Z) \mid U$ is row-standard mod $Y\}$. It is easy to see that $\{Z_U \in \Lambda_{\lambda/\mu}\varphi \mid U \in \mathrm{Row}_{\lambda/\mu}(Z, Y)\}$ is a basis of $\Lambda_{\lambda/\mu}\varphi$.

Now let us make these bases of $\Lambda_{\lambda/\mu}F$, $D_{\lambda/\mu}G$, and $\Lambda_{\lambda/\mu}\varphi$ ordered bases.

DEFINITION I.3.7. Let $W = \{w_1 < \cdots < w_k\}$ be a totally ordered set, and let $T \in \mathrm{Tab}_{\lambda/\mu}(W)$, and let $i, j \in \mathbb{N}$. Define $T_{i,j} = n_{[1,i]}(S, [w_1, w_j])$, where $[1, i] = \{1, ..., i\}$ and $[w_1, w_j] = \{w_1, ..., w_j\}$. If S is another tableau, we say $S \leqslant T$ if $S_{i,j} \geqslant T_{i,j}$ for every i, j. We say $S < T$ if $S \leqslant T$ and $S_{i,j} > T_{i,j}$ for at least one pair i, j.

This provides a pseudo-order on the set of tableaux. Observe that if we restrict this pseudo-order to the subset of row-standard tableaux, then on this subset the order is consistent with the usual lexicographic order. So $\geqslant$ makes these bases of $\Lambda_{\lambda/\mu}F$, $D_{\lambda/\mu}G$ and $\Lambda_{\lambda/\mu}\varphi$ into ordered bases, which need not be totally ordered.

DEFINITION I.3.8. Let A be a Hopf algebra (in G_R^2 or $\mathscr{C}$). We define $\square_A$ (resp. $\tilde{\square}_A$) to be the map given by

$$\square_A : A \otimes A \xrightarrow{\Delta_A \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes m_A} A \otimes A$$

$$\left(\text{resp. } \tilde{\square}_A : A \otimes A \xrightarrow{\mathrm{id} \otimes \Delta_A} A \otimes A \otimes A \xrightarrow{m_A \otimes \mathrm{id}} A \otimes A\right).$$

Let $\gamma, \nu \in \Omega^+$ with $\nu = \gamma + (\gamma_{t+1} - k) \cdot \alpha_t$ for some $t \in \mathbb{N}$ and $0 \leqslant k < \gamma_{t+1}$. We define $\square^\nu_\gamma : \Lambda_\nu\varphi \to \Lambda_\gamma\varphi$ and $\tilde{\square}^\gamma_\nu : \Lambda_\gamma\varphi \to \Lambda_\nu\varphi$ given by

$$\begin{array}{l} \square^\nu_\gamma : \Lambda_\nu\varphi = \Lambda^{\gamma_1}\varphi \otimes \cdots \otimes \Lambda^{\gamma_{t-1}}\varphi \otimes \Lambda^{\gamma_t + \gamma_{t+1} - k}\varphi \otimes \Lambda^k\varphi \otimes \Lambda^{\gamma_{t+2}}\varphi \otimes \cdots \otimes \Lambda^{\gamma_q}\varphi \\ \qquad\qquad \Big\downarrow \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \square_{\Lambda\varphi} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \\ \Lambda_\gamma\varphi = \Lambda^{\gamma_1}\varphi \otimes \cdots \otimes \Lambda^{\gamma_{t-1}}\varphi \otimes \Lambda^{\gamma_t}\varphi \otimes \Lambda^{\gamma_{t+1}}\varphi \otimes \Lambda^{\gamma_{t+2}}\varphi \otimes \cdots \otimes \Lambda^{\gamma_q}\varphi \end{array}$$

and

$$\begin{array}{l} \tilde{\square}^\gamma_\nu : \Lambda_\gamma\varphi = \Lambda^{\gamma_1}\varphi \otimes \cdots \otimes \Lambda^{\gamma_{t-1}}\varphi \otimes \Lambda^{\gamma_t}\varphi \otimes \Lambda^{\gamma_{t+1}}\varphi \otimes \Lambda^{\gamma_{t+2}}\varphi \otimes \cdots \otimes \Lambda^{\gamma_q}\varphi \\ \qquad\qquad \Big\downarrow \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \tilde{\square}_{\Lambda\varphi} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \\ \Lambda_\nu\varphi = \Lambda^{\gamma_1}\varphi \otimes \cdots \otimes \Lambda^{\gamma_{t-1}}\varphi \otimes \Lambda^{\gamma_t + \gamma_{t+1} - k}\varphi \otimes \Lambda^k\varphi \otimes \Lambda^k\varphi \otimes \Lambda^{\gamma_{t+2}}\varphi \otimes \cdots \otimes \Lambda^{\gamma_q}\varphi, \end{array}$$

respectively, where $q = \lg(\gamma)$, and $\square_{\Lambda\varphi}$ (resp. $\tilde{\square}_{\Lambda\varphi}$) is an appropriate box (resp. box tilde) map, namely, composition of diagonalization and multiplication (resp. multiplication and diagonalization). If $\lambda, \mu \in \Omega^+$, and $\gamma = \lambda - \mu$, then we may denote $\square^{\nu}_{\gamma}$ by $\square^{\nu}_{\lambda/\mu}$ and $\bar{\square}^{\gamma}_{\nu}$ by $\bar{\square}^{\lambda/\mu}_{\nu}$.

Let $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$. We define $\square_{\lambda/\mu}(\Lambda\varphi)$: $\sum_{\nu \in S_{\square}(\lambda/\mu)} \Lambda_{\nu}\varphi \to \Lambda_{\lambda/\mu}\varphi$ to be the sum of the maps $\square^{\nu}_{\lambda/\mu}: \Lambda_{\nu}\varphi \to \Lambda_{\lambda/\mu}\varphi$. We may consider that $\Lambda F = \Lambda F \otimes R \subset \Lambda F \otimes DG = \Lambda\varphi$ and $DG \subset \Lambda\varphi$. So this definition can be used to ΛF and DG via the restriction.

LEMMA I.3.9. *Let $\gamma_1, \gamma_2, u, v \in \mathbb{N}_0$ with $\gamma_1 > u$ and $\gamma_2 > u + v$. Denote by $\bar{\square}_u$ the composite map*

$$\Lambda^u\varphi \otimes \Lambda^{\gamma_1+\gamma_2-u-v}\varphi \otimes \Lambda^v\varphi \xrightarrow{\mathrm{id}\otimes\Delta\otimes\mathrm{id}} \Lambda^u\varphi \otimes \Lambda^{\gamma_1-u}\varphi \otimes \Lambda^{\gamma_2-v}\varphi \otimes \Lambda^v\varphi$$
$$\xrightarrow{m\otimes m} \Lambda^{\gamma_1}\varphi \otimes \Lambda^{\gamma_2}\varphi.$$

If $i_1, i_2 \in \mathbb{N}_0$ and T is a row standard tableau in $Z_{(u, \gamma_1+\gamma_2-u-v, v)}$ with $n_1(T, Y) = i_1$ and $n_2(T, Y) = i_2$, then $\bar{\square}_u(Z_T) \in \square_{(\gamma_1, \gamma_2)}(\sum_{k=0}^{u}\sum_{j=0}^{i_2+r_k} \Lambda^{\gamma_1+\gamma_2-k-j-v}F \otimes D_jG \otimes \Lambda^{k+v}\varphi)$, where $r_k = \min(i_1, u-k)$ for each k.

Proof. The proof is quite similar to the proof of [2, Lemma II.2.9], so we omit it.

Before stating the standard basis theorem, we introduce the notion of universally free functor.

DEFINITION I.3.10. Let $T_R(F_1, ..., F_n)$ be a functor to the category of R-modules defined for all commutative rings R and all n-tuples of (finite) free R-modules $F_1, ..., F_n$. T_R is called *universal* if $T_S(S\otimes -, ..., S\otimes -)$ is naturally equivalent to $S\otimes T_R(-, ..., -)$ for any ring morphism $\phi: R \to S$. T_R is called *universally free* if T_R is universal and $T_R(F_1, ..., F_n)$ is free for any n-tuple $F_1, ..., F_n$. Let $f_R: T_R \to T'_R$ be a natural transformation of universal functors defined for all commutative rings R. f_R is called *universal* if for any ring morphism $\phi: R \to S$ and any n-tuple of free R-modules $F_1, ..., F_n$ the diagram

$$\begin{array}{ccc} S\otimes T_R(F_1, ..., F_n) & \xrightarrow{S\otimes f_R(F_1, ..., F_n)} & S\otimes T'_R(F_1, ..., F_n) \\ \downarrow \simeq & & \downarrow \simeq \\ T_S(S\otimes F_1, ..., S\otimes F_n) & \xrightarrow{f_S(S\otimes F_1, ..., S\otimes F_n)} & T'_S(S\otimes F_1, ..., S\otimes F_n) \end{array}$$

is commutative.

For example, SF, ΛF, and DF are universally free. Their structure morphisms as Hopf algebras are universal. Tensor product and direct sum of universally free functors are universally free. If $f_R: T_R \to T'_R$ is universal, then $\mathrm{Cok}\, f_R$ is universal functor and $\mathrm{cok}\, f_R\colon T'_R \to \mathrm{Cok}\, f_R$ is universal, since $S \otimes -$ preserves cokernel.

THEOREM I.3.11 [2, Theorem V.1.10]. *Let $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$, and let φ, X, Y, and Z be as above.* $\{d_{\lambda/\mu}(Z_T) \mid T$ *is standard tableau* mod Y *in* $\mathrm{Tab}_{\lambda/\mu}(Z)\}$ *is a free basis for $L_{\lambda/\mu}\varphi$, and the sequence*

$$\sum_{\nu \in S_\square(\lambda/\mu)} \Lambda_\nu \varphi \xrightarrow{\square_{\lambda/\mu}} \Lambda_{\lambda/\mu}\varphi \xrightarrow{d_{\lambda/\mu}(\varphi)} L_{\lambda/\mu}\varphi \to 0$$

is exact. Hence (underlying module of) $L_{\lambda/\mu}\varphi$ is universally free and $d_{\lambda/\mu}(\varphi)$ is universal. In particular, $\{d_{\lambda/\mu}(X_T) \mid T$ *is a standard tableau in* $\mathrm{Tab}_{\lambda/\mu}(X)\}$ (resp. $\{d'_{\lambda/\mu}(Y_T) \mid T$ *is a co-standard tableau in* $\mathrm{Tab}_{\lambda/\mu}(Y)\}$) *is a free basis of* $L_{\lambda/\mu}F$ (resp. $K_{\lambda/\mu}G$), *and the sequence*

$$\sum_{\nu \in S_\square(\lambda/\mu)} \Lambda_\nu F \xrightarrow{\square_{\lambda/\mu}} \Lambda_{\lambda/\mu}F \xrightarrow{d_{\lambda/\mu}(F)} L_{\lambda/\mu}F \to 0$$

$$\left(\text{resp. } \sum_{\nu \in S_\square(\lambda/\mu)} D_\nu G \xrightarrow{\square_{\lambda/\mu}} D_{\lambda/\mu}G \xrightarrow{d'_{\lambda/\mu}(G)} K_{\lambda/\mu}G \to 0\right)$$

is exact. Hence, $L_{\lambda/\mu}F$ and $K_{\lambda/\mu}G$ are universally free.

LEMMA I.3.12. *Let $f_R\colon T_R \to T'_R$ be a universal natural transformation between universally free functors. If $F_1, ..., F_n$ is an n-tuple of free a R-module such that both* $\mathrm{rank}(T_{\mathbb{Z}}(F_1, ..., F_n))$ *and rank*$(T'_{\mathbb{Z}}(F_1, ..., F_n))$ *are finite, then the following conditions are equivalent*:

(i) Cok $T_{\mathbb{Z}}(F_1, ..., F_n)$ *is $\mathbb{Z}$-free.*

(ii) $\dim_{\mathbb{F}_p}(\ker T_{\mathbb{F}_p}(F_1, ..., F_n))$ *is independent of p, where $\mathbb{F}_p$ is the prime field of characteristic p.*

(iii) Cok T_R, Im T_R, *and* Ker T_R *are all universally free functors, and the natural morphisms* $T'_R \to \mathrm{Cok}\, f_R$, $\mathrm{Im}\, f_R \to T'_R$, *and* $\mathrm{Ker}\, f_R \to T_R$ *are all universal.*

Proof. Easy.

Let $F_1, ..., F_n$ be free $\mathbb{Z}$-modules, and let T_R be a functor from the full subcategory of n-tuples of R-modules whose object is only $(F_1^R, ..., F_n^R)$, where $F_i^R = R \otimes_{\mathbb{Z}} F_i$ for $1 \leqslant i \leqslant n$. We can define universality and universally freeness for such T_R in a way similar to (I.3.10). Such a "universally free functor" for a fixed n-tuple is called a *universally free* $GL(F_1^R) \times \cdots \times GL(F_n^R)$*-module.* We also say that T_R is universally free in this situation. Clearly, a fact similar to (I.3.12) holds in this case, too.

4. *Symmetric Functions*

The theory of symmetric functions is deeply related to the representation theory of general linear groups. In this section, we prepare some formulas of symmetric functions used later after [18].

Let $x = x_1, ..., x_n, ...$ be an infinite sequence of indeterminates. For $n \in \mathbb{N}_0$, we denote by R_n the polynomial ring $\mathbb{Z}[x_1, ..., x_n]$. The symmetric group $\mathfrak{S}_n$ acts on R_n via the permutations of indeterminates. We denote by Λ_n the invariant subring $R_n^{\mathfrak{S}_n}$. Λ_n is decomposed into direct sums: $\Lambda_n = \sum_{k \geqslant 0} \Lambda_n^k$, where Λ_n^k is the module of homogeneous invariants of degree k. For integers m, n with $m \geqslant n \geqslant 0$, we define $\rho_{n,m}: R_m \to R_n$ to be the map given by $\rho_{n,m}(x_i) = x_i$ $(i \leqslant n)$ and $\rho_{n,m}(x_i) = 0$ $(i > n)$. It is easy to see that $\rho_{n,m}(\Lambda_m^k) = \Lambda_n^k$. We define $\Lambda^k = \varprojlim_n \Lambda_n^k$, and $\Lambda = \sum_{k \geqslant 0} \Lambda^k$. $\Lambda_x = \Lambda$ has the structure of a graded ring. We call an element of Λ *a symmetric function* (*of* x). Λ is not dependent on the order of the sequence x. It only depends on x as a set.

DEFINITION I.4.1. Let $r \in \mathbb{N}_0$. We define e_r and h_r to be the symmetric functions given by

$$e_r = e_r(x) = \sum_{\lambda \in \Omega_r^+, \tilde{\lambda}_1 = r} x^\lambda \left(= \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_r} \right) \in \Lambda^r$$

$$h_r = h_r(x) = \sum_{\lambda \in \Omega_r^+} x^\lambda \in \Lambda^r,$$

where $x^\lambda = \prod_{i \in \mathbb{N}} x_i^{\lambda_i}$ (we use this notation later, too). Furthermore, for a partition λ, we define $s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leqslant i,j \leqslant \lg(\lambda)}$ ($h_l = 0$ for $l \leqslant 0$) and call s_λ *the Schur function corresponding to the partition* λ.

LEMMA I.4.2. [18, I.(2.4)]. *$\Lambda = \mathbb{Z}[e_1, e_2, ...]$ and the e_i are algebraically independent over $\mathbb{Z}$.*

By I.4.2, we can define a ring homomorphism $\omega (= \omega_x): \Lambda_x \to \Lambda_x$ given by $\omega(e_r) = h_r$.

LEMMA I.4.3 [18, I.(2.7)]. *ω is an involution (i.e., $\omega^2 = \mathrm{id}_\Lambda$).*

LEMMA I.4.4 [18, I.(3.3)]. *$\{s_\lambda \mid \lambda \in \Omega_k^-\}$ is a $\mathbb{Z}$-basis of Λ^k.*

Now let $\mu \in \Omega_k^-$ and $\nu \in \Omega_l^-$ $(k, l \in \mathbb{N}_0)$. By I.4.4, we can write

$$s_\mu \cdot s_\nu = \sum_{\lambda \in \Omega^-} c_{\mu,\nu}^\lambda \cdot s_\lambda \qquad (c_{\mu,\nu}^\lambda \in \mathbb{Z}).$$

Since $s_\mu \cdot s_\nu \in \Lambda^{k+l}$, $|\lambda| \neq k+l$ implies $c^\lambda_{\mu,\nu}=0$. For arbitrary partitions λ and μ, we define

$$s_{\lambda/\mu} = \sum_{\nu \in \Omega^-} c^\lambda_{\mu,\nu} \cdot s_\nu$$

and call $s_{\lambda/\mu}$ a *skew Schur function corresponding to* λ *and* μ. Then we have

LEMMA I.4.5 [18, I.(5.4), (5.5), *and* (5.6)].

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{1 \leqslant i,j \leqslant \tilde{\lambda}_1} = \det(e_{\tilde{\lambda}_i-\tilde{\mu}_j-i+j})_{1 \leqslant i,j \leqslant \lambda_1}$$

so that $\omega(s_{\lambda/\mu}) = s_{\tilde{\lambda}/\tilde{\mu}}$.

In particular, for $k, l \in \mathbb{N}_0$ with $k \geqslant l$, it holds that $s_{k\cdot\varepsilon_1/l\cdot\varepsilon_1} = h_{k-l}$ and that $s_{\omega_k/\omega_l} = e_{k-l}$. Note that $s_{\lambda/0} = s_\lambda$ for any partition λ.

Now we consider two sets of indeterminates $x = x_1, x_2, \ldots$ and $y = y_1, y_2, \ldots$. We denote by (x, y) the set of indeterminates $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$.

LEMMA I.4.6 [18, I.(5.9)]. *If* $\lambda \in \Omega^-$, *then*

$$s_\lambda(x, y) = \sum_{\mu \subset \lambda, \mu \in \Omega^-} s_{\lambda/\mu}(x) \cdot s_\mu(y).$$

In particular, for $r \in \mathbb{N}_0$, *it holds that*

$$h_r(x, y) = \sum_{k+l=r} h_k(x) \cdot h_l(y).$$

LEMMA I.4.7. *For* $k \in \mathbb{N}_0$, *it holds that*

$$h_k(x \cdot y) = \sum_{\lambda \in \Omega^-_k} s_\lambda(x) \cdot s_\lambda(y),$$

where $h_k(x \cdot y)$ *stands for the image of* $h_k(z)$ *(*z *is another set of variables;* $z = \{z_{11}, z_{12}, \ldots, z_{21}, z_{22}, \ldots\}$*) by the ring homomorphism* $\Lambda_z \to \Lambda_x \otimes \Lambda_y$ *given by* $z_{ij} \mapsto x_i \cdot y_j$ *(we use this notation later, too).*

Proof. By [18,I.(4.3), it holds that

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \Omega^-} s_\lambda(x) \cdot s_\lambda(y).$$

Since $(1 - x_i y_j)^{-1} = 1 + x_i y_j + x_i^2 y_j^2 + \cdots$, we can get the formula comparing the homogeneous part of degree $2 \cdot k$ of both sides. Q.E.D.

Similarly, we can prove the next lemma, from the equality $\prod_{i,j} (1 + x_i y_j)^{-1} = \sum_{\lambda \in \Omega^-} s_\lambda(x) \cdot s_{\tilde{\lambda}}(y)$ (see [18, I.(4.3′)]).

LEMMA I.4.8. *Let k be as above. We have* $e_k(x \cdot y) = \sum_{\lambda \in \Omega_k^-} s_\lambda(x) \cdot s_{\tilde{\lambda}}(y)$.

Now we consider the four sets of indeterminates $x = x_1, x_2, \ldots$, $\xi = \xi_1, \xi_2, \ldots$, $y = y_1, y_2, \ldots$, and $\eta = \eta_1, \eta_2, \ldots$. Abusing the notation, we simply denote the involution $\mathrm{id}_{\Lambda_x} \otimes \omega_\xi \otimes \mathrm{id}_{\Lambda_y} \otimes \mathrm{id}_{\Lambda_\eta}: \Lambda_x \otimes \Lambda_\xi \otimes \Lambda_y \otimes \Lambda_\eta \to \Lambda_x \otimes \Lambda_\xi \otimes \Lambda_y \otimes \Lambda_\eta$ by ω_ξ. We also use the abused notation ω_η.

PROPOSITION I.4.9. *Let $k \in \mathbb{N}_0$. We have*

$$\sum_{\alpha+\beta+\gamma+\delta=k} h_\alpha(x \cdot y) \cdot e_\beta(x \cdot \eta) \cdot e_\gamma(\xi \cdot y) \cdot h_\delta(\xi \cdot \eta)$$
$$= \sum_{\lambda \in \Omega_k^-} \sum_{\substack{\mu, \nu \subset \lambda \\ \mu, \nu \in \Omega^-}} s_{\lambda/\mu}(x) \cdot s_{\tilde{\mu}}(\xi) \cdot s_{\lambda/\nu}(y) \cdot s_{\tilde{\nu}}(\eta).$$

Proof. By I.4.7 and I.4.8, for any $\beta \in \mathbb{N}_0$, we have

$$\omega_\eta e_\beta(x \cdot \eta) = \omega_\eta \left(\sum_{\lambda \in \Omega_\beta^-} s_\lambda(x) \cdot s_{\tilde{\lambda}}(\eta) \right)$$
$$= \sum_{\lambda \in \Omega_\beta^-} s_\lambda(x) \cdot s_\lambda(\eta) = h_\beta(x \cdot \eta).$$

Similarly, for $\gamma, \delta \in \mathbb{N}_0$, we have that $\omega_\xi e_\gamma(\xi \cdot y) = h_\gamma(\xi \cdot y)$ and that $\omega_\xi \cdot \omega_\eta h_\delta(\xi \cdot \eta) = h_\delta(\xi \cdot \eta)$. So we have

$$\omega_\xi \cdot \omega_\eta \left(\sum_{\alpha+\beta+\gamma+\delta=k} h_\alpha(x \cdot y) \cdot e_\beta(x \cdot \eta) \cdot e_\gamma(\xi \cdot y) \cdot h_\delta(\xi \cdot \eta) \right)$$
$$= \sum_{\alpha+\beta+\gamma+\delta=k} h_\alpha(x \cdot y) \cdot h_\beta(x \cdot \eta) \cdot h_\gamma(\xi \cdot y) \cdot h_\delta(\xi \cdot \eta)$$
$$= \sum_{k_1+k_2=k} h_{k_1}(x \cdot y, x \cdot \eta) \cdot h_{k_2}(\xi \cdot y, \xi \cdot \eta)$$
$$= \sum_{k_1+k_2=k} h_{k_1}(x \cdot (y, \eta)) \cdot h_{k_2}(\xi \cdot (y, \eta))$$
$$= h_k(x \cdot (y, \eta), \xi \cdot (y, \eta)) = h_k((x, \xi) \cdot (y, \eta))$$
$$= \sum_{\lambda \in \Omega_k^-} s_\lambda(x, \xi) \cdot s_\lambda(y, \eta)$$
$$= \sum_{\lambda \in \Omega_k^-} \sum_{\mu, \nu \subset \lambda, \mu, \nu \in \Omega^-} s_{\lambda/\mu}(x) \cdot s_\mu(\xi) \cdot s_{\lambda/\nu}(y) \cdot s_\nu(\eta)$$
$$= \omega_\xi \cdot \omega_\eta \left(\sum_\lambda \sum_{\mu, \nu} s_{\lambda/\mu}(x) \cdot s_{\tilde{\mu}}(\xi) \cdot s_{\lambda/\nu}(y) \cdot s_{\tilde{\nu}}(\eta) \right).$$

By I.4.3, $\omega_\xi \cdot \omega_\eta$ is an isomorphism. So we have the desired formula.

Q.E.D.

Now let R be a commutative ring and F a finite free R-module of rank m. By [18, I.(5.12)] and I.3.11, we have the following formula.

LEMMA I.4.10. *Let $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$. We have*

$$\operatorname{rank} L_{\lambda/\mu}F = \operatorname{rank} K_{\tilde{\lambda}/\tilde{\mu}}F = s_{\tilde{\lambda}/\tilde{\mu}}(\mathbb{1}_m),$$

where $s_{\tilde{\lambda}/\tilde{\mu}}(\mathbb{1}_m)$ is the value of skew Schur function in m variables $s_{\tilde{\lambda}/\tilde{\mu}}(x_1, ..., x_m)$ at $x_1 = \cdots = x_m = 1$ (we use this notation later).

Proof. Each term of the equation is equal to the cardinality of the standard tableaux of shape λ/μ with values in an ordered basis of F. So we have the formula. Q.E.D.

Since $L_{k\cdot\varepsilon_1}F = \Lambda^k F$, $L_{\omega_k}F = S_k F$, and $K_{k\cdot\varepsilon_1}F = D_k F$, we have

$$\operatorname{rank} S_k F = \operatorname{rank} D_k F = h_k(\mathbb{1}_m) \qquad \text{and} \qquad \operatorname{rank} \Lambda^k F = e_k(\mathbb{1}_m)$$

for $k \in \mathbb{N}_0$.

Now let $\varphi: F_1 \to F_0$ be a morphism of finite free R-module. We put rank $F_i = m_i$ $(i = 0, 1)$. For partitions $\lambda, \mu \in \Omega^-$ with $\lambda \subset \mu$, the rank of the underlying module of $L_{\lambda/\mu}\varphi$ is calculated as follows (see [2, V.1.14]):

$$\begin{aligned}\operatorname{rank} L_{\lambda/\mu}\varphi &= \sum_{\mu \subset \gamma \subset \lambda} \operatorname{rank}[L_{\gamma/\mu}F_0 \otimes L_{\lambda/\gamma}F_1] \\ &= \sum_{\mu \subset \gamma \subset \lambda} s_{\lambda/\gamma}(\mathbb{1}_{m_1}) \cdot s_{\tilde{\gamma}/\tilde{\mu}}(\mathbb{1}_{m_0}).\end{aligned}$$

In particular, we have

$$\begin{aligned}\operatorname{rank} S_k\varphi &= \sum_{i+j=k} \operatorname{rank} S_i F_0 \otimes \Lambda^j F_1 = \sum_{i+j=k} h_i(\mathbb{1}_{m_0}) \cdot e_j(\mathbb{1}_{m_1}) \\ \operatorname{rank} \Lambda^k\varphi &= \sum_{i+j=k} \operatorname{rank} \Lambda^i F_0 \otimes D_j F_1 = \sum_{i+j=k} e_i(\mathbb{1}_{m_0}) \cdot h_j(\mathbb{1}_{m_1}),\end{aligned}$$

where k is a nonnegative integer.

LEMMA I.4.11. *Let $\varphi: F_1 \to F_0$ and $\psi: G_1 \to G_0$ be morphisms of finite free R-modules, so that $\varphi \otimes \psi$ is the complex*

$$0 \to F_1 \otimes G_1 \xrightarrow{-1\otimes\psi + \varphi\otimes 1} F_1 \otimes G_0 \oplus F_0 \otimes G_1 \xrightarrow{\varphi\otimes 1 + 1\otimes\psi} F_0 \otimes G_0 \to 0.$$

We put rank $F_i = m_i$ *and rank* $G_i = n_i$ *for* $i = 0, 1$. *Let* $k \in \mathbb{N}_0$. *We have*

$$\operatorname{rank} S_k(\varphi \otimes \psi) = \operatorname{rank}\left[\sum_{\lambda \in \Omega_k^-} L_\lambda\varphi \otimes L_\lambda\psi\right],$$

where $S_k(\varphi \otimes \psi)$ on the left hand side is the extended symmetric power defined in Sect. 2.

Proof. From (I.4.9) and (I.4.10), we have

$$\begin{aligned}
\operatorname{rank} S_k(\varphi \otimes \psi) &= \sum_{\alpha+\beta+\gamma+\delta=k} \operatorname{rank}[D_\alpha(F_1 \otimes G_1) \otimes \Lambda^\beta(F_1 \otimes G_0) \\
&\qquad \otimes \Lambda^\gamma(F_0 \otimes G_1) \otimes S_\delta(F_0 \otimes G_0)] \\
&= \sum_{\alpha+\beta+\gamma+\delta=k} h_\alpha(\mathbb{1}_{m_1 n_1}) \cdot e_\beta(\mathbb{1}_{m_1 n_0}) \cdot e_\gamma(\mathbb{1}_{m_0 n_1}) \cdot h_\delta(\mathbb{1}_{m_0 n_0}) \\
&= \sum_{\lambda \in \Omega_k^-} \sum_{\substack{\mu, \nu \in \Omega^- \\ \mu, \nu \subset \lambda}} s_{\lambda/\mu}(\mathbb{1}_{m_1}) \cdot s_{\tilde{\mu}}(\mathbb{1}_{m_0}) \cdot s_{\lambda/\nu}(\mathbb{1}_{n_1}) \cdot s_{\tilde{\nu}}(\mathbb{1}_{n_0}) \\
&= \sum_{\lambda \in \Omega_k^-} (\operatorname{rank} L_\lambda \varphi) \cdot (\operatorname{rank} L_\lambda \psi).
\end{aligned}$$

So the assertion is clear. Q.E.D.

II. Basic Facts on Determinantal Ideals

This chapter is devoted to introducing the basic facts on determinantal ideals. There is no new result in this chapter. Therefore almost all proofs are omitted.

Throughout this chapter, R is a Noetherian ring with unit. Let m, n be positive integers and denote by S the polynomial ring over R with $m \cdot n$ variables x_{ij} for $i = 1, ..., m$ and $j = 1, ..., n$. Then $S = \bigoplus_{r \geqslant 0} S_r$ is a graded ring with $\deg(x_{ij}) = 1$ for all i and j. Consider the $m \times n$ matrix (x_{ij}) with entries in S. We call such a matrix *a generator matrix*. For a positive integer t such that $1 \leqslant t \leqslant \min(m, n)$, I_t is defined to be the ideal of S generated by all t-minors of (x_{ij}). We call such ideals *determinantal ideals.* For a non-negative integer r we denote by $I_{t,r}$ the homogeneous component $I_t \cap S_r$. Then there exist the decompositions $I_t = \bigoplus_{r \geqslant 0} I_{t,r}$ and $S/I_t = \bigoplus_{r \geqslant 0} S_r/I_{t,r}$. We call such quotient rings *determinantal varieties.*

1. *Homological Properties*

This section is devoted to the homological properties of determinantal ideals. As for the proofs of theorems, we refer to Hochster and Eagon [15] and Svanes [26].

THEOREM II.1.1 [15]. *Let m, n, and t be positive integers such that $1 \leqslant t \leqslant \min(m, n)$. Then,*

(1) proj. $\dim_S(S/I_t) = \text{grade}(I_t) = (m-t+1)(n-t+1)$

(2) *if R is a (normal) domain, then so is S/I_t.*

Theorem II.1.1 implies that I_t is perfect for any m, n, and t. Therefore S/I_t is Cohen–Macaulay when the coefficient ring R is Cohen–Macaulay.

Remark II.1.2. Let R be the integers $\mathbb{Z}$. By Theorem II.1.1, S/I_t is a torsion free $\mathbb{Z}$-module. Therefore for any positive integer r, $S_r/I_{t,r}$ is a free $\mathbb{Z}$-module. By this fact, it is easy to see that over an arbitrary commutative ring R, $S_r/I_{t,r}$ is a finitely generated free R-module and $\operatorname{rank}_R(S_r/I_{t,r})$ does not depend on R.

DEFINITION II.1.3. Let $A = \bigoplus_{r \geqslant 0} A_r$ be a Noetherian graded ring such that A_0 is a field, and $M = \bigoplus_{r \geqslant q} M_r$ be a finitely generated graded A-module. Then define the *Poincaré series* $F(M, \lambda)$ of M to be the formal power series

$$F(M, \lambda) = \sum_{r=q}^{\infty} (\dim_{A_0} M_r) \cdot \lambda^r \in \mathbb{Z}[[\lambda]].$$

(When $q = 0$, it is well known that we can write $F(M, \lambda)$ in the form

$$F(M, \lambda) = \frac{P(M, \lambda)}{\prod_{i=1}^{d} (1 - \lambda^{e_i})},$$

where $P(M, \lambda)$ is a polynomial in λ with coefficients in $\mathbb{Z}$.)

Remark II.1.4. Let R be a field. Then by Remark II.1.2, the Poincaré series $F(S/I_t, \lambda)$ does not depend on the coefficient field.

THEOREM II.1.5 [26]. *Let R be a field. Then S/I_t is Gorenstein if and only if $m = n$ or $t = 1$.*

Our main object in this paper is the determinantal ideals such that $m = n = t + 2$. They are Gorenstein by Theorem II.1.5.

2. *Canonical Modules*

Let $A = \bigoplus_{r \geqslant 0} A_r$ be a Noetherian graded ring. We denote by $\mathfrak{m}$ the homogeneous ideal $\bigoplus_{r > 0} A_r$.

Let M, N be graded A-modules and let $a \in \mathbb{Z}$. We denote by $N(a)$ the graded A-module which coincides with N as the underlying A-module and whose grading is given by $[N(a)]_b = N_{a+b}$ for all $b \in \mathbb{Z}$. Let $\operatorname{Hom}_A(M, N)_a$ denote the abelian group of all the graded homomorphisms from M to $N(a)$. We put $\underline{\operatorname{Hom}}_A(M, N) = \bigoplus_{a \in \mathbb{Z}} \operatorname{Hom}_A(M, N)_a$ and consider it as a graded A-module with $\{\operatorname{Hom}_A(M, N)_a\}_{a \in \mathbb{Z}}$ as its grading. The derived functors of $\underline{\mathrm{Hom}}_A(\ ,\ )$ are denoted by $\underline{\mathrm{Ext}}^i_A(\ ,\ )$. Since A is Noetherian, $\underline{\mathrm{Ext}}^i_A(M, N) = \mathrm{Ext}^i_A(M, N)$ as the underlying A-module if M is a finitely generated graded A-module. Further $M \underline{\otimes} N$ is defined to be the graded module $\bigoplus_{a \in \mathbb{Z}} (M \underline{\otimes} N)_a$, where $(M \underline{\otimes} N)_a$ is a A_0-submodule of $M \otimes_A N$ generated by $\bigoplus_{i+j=a} M_i \otimes_{A_0} N_j$. (Remember that there exists a canonical surjection $M \otimes_{A_0} N \to M \otimes_A N$.) The derived functors of $\underline{\otimes}$ is denoted by $\underline{\mathrm{Tor}}^A_i(\ ,\ )$. By definition $\underline{\mathrm{Tor}}^A_i(\ ,\ )$ is isomorphic to $\mathrm{Tor}^A_i(\ ,\ )$ as the underlying A-module. If there is no possibility of confusion, we denote $\underline{\otimes}$ or $\underline{\mathrm{Tor}}^A_i(\ ,\ )$ simply by $\otimes$ or $\mathrm{Tor}^A_i(\ ,\ )$.

In the rest of this section we assume that A_0 is a field k.

Definition II.2.1. For every integer $i \geqslant 0$, we put

$$\underline{H}^i_{\mathfrak{m}}(\ ) = \varinjlim_{s} \underline{\mathrm{Ext}}^i_A(A/\mathfrak{m}^s,\ )$$

and call it *the* ith *local cohomology functor.*

Definition II.2.2. If $\dim(A) = d$, we put

$$K_A = \underline{\mathrm{Hom}}_k(\underline{H}^d_{\mathfrak{m}}(A), k),$$

the k-dual of the dth local cohomology group of A. As $\underline{H}^i_{\mathfrak{m}}(A)$ is an Artinian graded A-module, K_A is a finitely generated graded A-module. We call K_A the *canonical module* of A. As for the canonical modules, the following three propositions are well known.

Proposition II.2.3 [11, Proposition (2.1.3)]. *If A is a Cohen–Macaulay ring, then A is Gorenstein if and only if $K_A \simeq A(q)$ for some $q \in \mathbb{Z}$.*

Let $T = \bigoplus_{r \geqslant 0} T_r$ be also a Noetherian graded ring such that T_0 is k.

Proposition II.2.4 [11, Proposition (2.2.9)]. *Let $f: T \to A$ be a homomorphism of graded k-algebras. We assume that T is Cohen–Macaulay and that A is a finite T-module. (We do not assume that f is injective.) If we put $e = \dim(T) - \dim(A)$, then*

$$K_A = \underline{\mathrm{Ext}}^e_T(A, K_T).$$

Proposition II.2.5 [25, Theorem 4.4]. *Let A be Cohen–Macaulay and $\dim(A) = d$; then*

$$F(K_A, \lambda) = (-1)^d F\left(A, \frac{1}{\lambda}\right).$$

Remark II.2.6. Suppose $m = n$ and R is a field k. Since S/I_t is

Gorenstein (Theorem II.1.5), $K_{S/I_t} = (S/I_t)(\rho)$ for some integer $\rho \in \mathbb{Z}$. Then ρ does not depend on the coefficient field k, because the Poincaré series $F(K_{S/I_t}, \lambda)$ is independent of k by Remark II.1.4 and Proposition II.2.5.

3. *Minimal Free Resolutions and Betti Numbers*

DEFINITION II.3.1. Let $\mathbb{P}.$ be a graded S-free resolution of S/I_t. We say that $\mathbb{P}.$ is *minimal* when all entries of all boundary maps of $\mathbb{P}.$ are contained in I_1.

When R is a field, it is well known that there exists a minimal free resolution of S/I_t (for instance, see [20]). In the late 1970s Lascoux [17] gave an explicit description of minimal free resolutions of S/I_t for any m, n, and t, when R contains the field of rational numbers $\mathbb{Q}$ (he used the standard classical representation theory of general linear groups). But over an arbitrary commutative ring R, the existence of minimal free resolutions is not known in general. (If $t = \min(m, n)$, such resolutions were constructed by Eagon and Northcott [9], Buchsbaum, and Rim, separately. If $m = n = t + 1$, the Gulliksen–Negård complex [13] is so. More generally Akin, Buchsbaum, and Weyman [1] constructed minimal free resolutions when $t + 1 = \min(m, n)$. It is easy to see that we can construct minimal free resolutions if $t = 1$ by the Koszul complex. Recently Hashimoto [14] has shown that there do not exist minimal free resolutions in general.)

DEFINITION II.3.2. For a non-negative integer r, $\mathrm{Tor}_r^S(S/I_t, S/I_1)$ is a finitely generated R-module. When R is a domain, we call $\mathrm{rank}_R(\mathrm{Tor}_r^S(S/I_t, S/I_1))$ the rth *Betti number* of S/I_t.

Remark II.3.3. By Theorem II.1.1, if there exists a minimal free resolution when R is the integers $\mathbb{Z}$, then we can construct minimal free resolutions over an arbitrary commutative ring R.

Next we give the equivalent condition for existence of minimal free resolutions. Let R be the integers $\mathbb{Z}$. We denote by $\mathbb{F}_p$ the prime field of characteristic $p > 0$.

PROPOSITION II.3.4 [24, Proposition 2 of Chap. 4]. *For any fixed integers m, n, t and i satisfying $i \geq 0$ and $1 \leq t \leq \min(m, n)$, the following conditions are equivalent*:

(1) *There exists a graded free complex* $P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} \cdots \to P_1 \xrightarrow{\partial_1} P_0$ *such that* $P_{i+1} \to \cdots \to P_1 \to P_0 \xrightarrow{\partial_0} S/I_t \to 0$ *is exact for some graded homomorphism ∂_0 and the entries of $\partial_1, \ldots, \partial_{i+1}$ are contained in I_1.*

(2) *For any integer q satisfying $0 \leqslant q \leqslant i$, $\dim_{\mathbb{F}_p} \mathrm{Tor}_q^{S \otimes \mathbb{F}_p}((S/I_t) \otimes \mathbb{F}_p, (S/I_1) \otimes \mathbb{F}_p)$ does not depend on p. (Tensor products above are defined over $\mathbb{Z}$.)*

(3) *For any integer q satisfying $0 \leqslant q \leqslant i$, $\mathrm{Tor}_q^S(S/I_t, S/I_1)$ is a $\mathbb{Z}$-free module.*

Proof. $(1) \Rightarrow (2)$ is obvious by Theorem II.1.1.

We now show that $(2) \Rightarrow (3)$. Suppose that there exists q such that $0 \leqslant q \leqslant i$ and $\mathrm{Tor}_q^S(S/I_t, S/I_1)$ is not $\mathbb{Z}$-free. We assume that $\mathrm{Tor}_r^S(S/I_t, S/I_1)$ is $\mathbb{Z}$-free for $0 \leqslant r < q$. Let $\mathbb{G}.$ be a graded finite free resolution of S/I_t. We denote by $\mathbb{Q}.$ the $\mathbb{Z}$-free complex $\mathbb{G}. \otimes_S (S/I_1)$. (Remember that each Q_j is a finite free $\mathbb{Z}$-module.) Then it is easy to see that $\mathrm{Tor}_j^S(S/I_t, S/I_1) = H_j(\mathbb{Q}.)$ and $\mathrm{Tor}_j^{S \otimes \mathbb{F}_p}((S/I_t) \otimes \mathbb{F}_p, (S/I_1) \otimes \mathbb{F}_p) = H_j(\mathbb{Q}. \otimes \mathbb{F}_p)$ for any integer j, because $\mathbb{G}. \otimes \mathbb{F}_p$ is a graded free resolution of $(S/I_t) \otimes \mathbb{F}_p$ by Theorem II.1.1.

At first suppose $q = 0$. By the argument above we have $H_0(\mathbb{Q}.) \otimes \mathbb{F}_p = H_0(\mathbb{Q}. \otimes \mathbb{F}_p)$. By our assumption $\dim_{\mathbb{F}_p} H_0(\mathbb{Q}. \otimes \mathbb{F}_p)$ does not depend on p. This implies that $H_0(\mathbb{Q}.)$ is $\mathbb{Z}$-free. Contradiction.

Second suppose $q > 0$. We define $Z_j = \ker(Q_j \to Q_{j-1})$ for each positive integer j, and $Z_0 = Q_0$. Since $H_r(\mathbb{Q}.)$ is $\mathbb{Z}$-free for $1 \leqslant r < q$, $0 \to Z_q \otimes \mathbb{F}_p \to Q_q \otimes \mathbb{F}_p \to Q_{q-1} \otimes \mathbb{F}_p$ is exact for any p. Therefore $H_q(\mathbb{Q}. \otimes \mathbb{F}_p)$ is equal to $H_q(\mathbb{Q}.) \otimes \mathbb{F}_p$ for any p. Hence $H_q(\mathbb{Q}.)$ is $\mathbb{Z}$-free. Contradiction.

Next we show $(3) \Rightarrow (1)$ by induction on i. Suppose $i = 0$. Then, it is easy to see that there exists such a complex. Suppose $i > 0$ and $P_i \to P_{i-1} \to \cdots \to P_1 \to P_0$ is a complex such that the condition of (1) is satisfied and $\mathrm{Tor}_i^S(S/I_t, S/I_1)$ is $\mathbb{Z}$-free. Choose $\mathbb{P}.$ such that $\mathrm{rank}_S(P_i)$ is minimum. Let F_{i+1} be a finite graded S-free module such that $F_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} P_{i-1}$ is a graded exist sequence. Then

$$F_{i+1}/I_1 \cdot F_{i+1} \xrightarrow{\bar{\partial}_{i+1}} P_i/I_1 \cdot P_i \to \mathrm{Tor}_i^S(S/I_t, S/I_1) \to 0$$

is exact ($\bar{\partial}_{i+1}$ is the morphism induced by ∂_{i+1}). Let K be $\mathrm{Im}(\bar{\partial}_{i+1})$. Obviously K is a graded $\mathbb{Z}$-free module. By our assumption of the minimality of $\mathrm{rank}(P_i)$, it is easy to check that $K = 0$. Therefore $F_{i+1} \to P_i \to \cdots \to P_0$ is a complex such that the condition of (1) is satisfied. Q.E.D.

Remark II.3.5. In order to show the existence of a minimal free resolution of S/I_t, we have only to show that all Betti numbers do not depend on p.

4. *Depth Sensitivity*

This section is devoted to depth sensitivity. For details we refer to [4]. Let A be a Noetherian commutative ring with unit.

PROPOSITION II.4.1 [6, Proposition 2.1]. *Let I be an ideal of A and suppose that we have a cohomology functor $\{T^i\}$ from the category of finitely generated A-modules into itself, such that*

(1) $T^i(r) = r$ *for every homothety r in A*
(2) $T^0(M) = 0$ *if and only if* $\operatorname{supp}(A/I) \cap \operatorname{Ass}(M) = \varnothing$
(3) $\operatorname{Supp}(T^i(M)) \subseteq \operatorname{Supp}(A/I)$.

Then for each A-module M such that $M/I \cdot M \neq 0$, $\operatorname{depth}(I, M)$ *is the smallest integer d such that $T^d(M) \neq 0$.*

Remark II.4.2. Let $\mathbb{P}.$ be a graded free resolution of S/I_t. We define $T^i(-) = H_{(m-t+1)(n-t+1)-i}(\mathbb{P}. \otimes -)$. See Proposition II.4.4, replacing I by I_t, and A by S. By the perfectness of I_t (Theorem II.1.1) and the Acyclic Lemma [5], it is easy to check that our cohomology functor $H_{(m-t+1)(n-t+1)-i}(\mathbb{P}. \otimes -)$ satisfies the three conditions of Proposition II.4.1. Hence for any finitely generated S-module M, we obtain the equality

$$\operatorname{depth}(I_t, M) = \min\{i \mid H_{(m-t+1)(n-t+1)-i}(\mathbb{P}. \otimes M) \neq 0\}.$$

We call such properties *depth sensitivity of resolutions of perfect ideals.*

5. *The First Syzygies of Determinantal Ideals*

This section is devoted to the first syzygies of determinantal ideals. For the proof of the theorem, we refer to [16].

It is easy to see that over an arbitrary coefficient ring R there exists a graded exact sequence

$$S(-t)^{\beta_1} \xrightarrow{\partial_1} S(0) \xrightarrow{\partial_0} S/I_t \longrightarrow 0,$$

where $\beta_1 = \binom{m}{t} \cdot \binom{n}{t}$ is the number of t-minors of the $m \times n$-matrix (x_{ij}), and ∂_0 is the projection and ∂_1 sends each basis element of $S(-t)^{\beta_1}$ to a t-minor of (x_{ij}).

THEOREM II.5.1 [16]. *For any integers m, n, and t, and an arbitrary* *commutative ring R, the relation module on I_t is generated by degree 1 relations on t-minors of (x_{ij}).*

In short, there exist β_2 and ∂_2 such that

$$S(-t-1)^{\beta_2} \xrightarrow{\partial_2} S(-t)^{\beta_1} \xrightarrow{\partial_1} S(0) \xrightarrow{\partial_0} S/I_t \longrightarrow 0 \cdots \tag{*}$$

is exact. (The entries of ∂_1 and ∂_2 are contained in I_1.)

Remark II.5.2. Let R be the integers $\mathbb{Z}$. Look at the degree $t+1$ component of the exact sequence (*),

$$S_0^{\beta_2} \xrightarrow{(\partial_2)_{t+1}} S_1^{\beta_1} \xrightarrow{(\partial_1)_{t+1}} S_{t+1} \xrightarrow{(\partial_0)_{t+1}} S_{t+1}/I_{t,t+1} \to 0.$$

Since $S_{t+1}/I_{t,t+1}$ is $\mathbb{Z}$-free, so is $\mathrm{Ker}((\partial_1)_{t+1})$. Let $r = \mathrm{rank}_{\mathbb{Z}}(\mathrm{Ker}((\partial_1)_{t+1}))$. Then we have another exact sequence

$$S(-t-1)^r \xrightarrow{\partial_2'} S(-t)^{\beta_1} \xrightarrow{\partial_1} S(0) \xrightarrow{\partial_0} S/I_t \to 0$$

whose degree $t+1$ component is the following exact sequence:

$$0 \to S_0^r \xrightarrow{(\partial_2')_{t+1}} S_1^{\beta_1} \xrightarrow{(\partial_1)_{t+1}} S_{t+1} \xrightarrow{(\partial_0)_{t+1}} S_{t+1}/I_{t,t+1} \to 0.$$

When R is a field k, the exact sequences above implies that $r = \dim_k(\mathrm{Tor}_2^S(S/I_t, S/I_1))$. Therefore the second Betti number does not depend on $\mathrm{ch}(k)$ for any m, n, and t.

6. *Linear Complexes*

Assume that there exists a minimal free resolution $\mathbb{P}.$ of S/I_t for certain m, n, and t over a certain coefficient ring R. It is easy to see that $P_0 = S(0)$ and $P_1 = S(-t)^{\binom{m}{t} \cdot \binom{n}{t}}$. Since $\mathbb{P}.$ is minimal, there exist integers β_{ij} such that P_i is written in the form $\bigoplus_{j \geqslant i+t-1} S(-j)^{\beta_{ij}}$ for each $i > 0$. We put $X_i = S(-i-t+1)^{\beta_{i,i+t-1}}$ for each positive integer i. By definition X_i is a direct summand of P_i. Let ∂_i's be boundary maps of $\mathbb{P}.$, where $P_{i+1} \xrightarrow{\partial_{i+1}} P_i$. Then by the minimality of $\mathbb{P}.$, $\partial_{i+1}(X_{i+1}) \subset X_i$ is satisfied for any positive integer i. Therefore $\{X_i\}_{i>0}$ is a subcomplex of $\mathbb{P}.$. We call this subcomplex *the linear complex*. We denote by δ_i the restriction of ∂_i to X_i.

Remark II.6.1. By the minimality of $\mathbb{P}.$ we obtain the exact sequences

$$\begin{array}{ccccccc} & & (X_{i+1})_{i+t} & & (X_i)_{i+t} & & (X_{i-1})_{i+t} \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & S_0^{\beta_{i+1,i+t}} & \xrightarrow{(\delta_{i+1})_{i+t}} & S_1^{\beta_{i,i+t-1}} & \xrightarrow{(\delta_i)_{i+t}} & S_2^{\beta_{i-1,i+t-2}} \end{array}$$

for each $i \geqslant 2$.

DEFINITION II.6.2 [1, Remark 3.19]. Let F and G be finitely generated free R-modules of rank m and n, respectively. For a positive integer i, we define a graded S-module $X_i^t(m, n)$ to be the kernel of the map

$$S(-i-t+1) \otimes \Lambda^{i-1}(F \otimes G) \otimes \Lambda^t F \otimes \Lambda^t G$$
$$\xrightarrow{\xi_{i-1}^t} S(-i-t+1) \otimes \Lambda^{i-2}(F \otimes G) \otimes S_{t+1}(F \otimes G),$$

(the above map is induced by the composition

$$\Lambda^{i-1}(F \otimes G) \otimes \Lambda^t F \otimes \Lambda^t G \xrightarrow{1 \otimes \alpha} \Lambda^{i-1}(F \otimes G) \otimes S_t(F \otimes G)$$
$$\xrightarrow{\partial} \Lambda^{i-2}(F \times G) \otimes S_{t+1}(F \otimes G),$$

where α is the canonical embedding $\Lambda^t F \otimes \Lambda^t G \to S_t(F \otimes G)$ (see Sect. 1 of Chapter III) and ∂ is the Koszul map, that is, the differential of the Koszul complex $\Lambda(F \otimes G) \otimes S(F \otimes G)$, and all tensor product are defined over R). If no confusion is possible, we denote $X_i^t(m, n)$ simply by X_i^t.

Furthermore, for a positive integer i, a graded S-module $Z_i^t(m, n)$ is defined to be the kernel of the map

$$S(-i-t+1) \otimes \Lambda^{i-1}(F \otimes G) \otimes \Lambda^t F \otimes \Lambda^t G$$
$$\xrightarrow{\eta_{i-1}^t} S(-i-t+1) \otimes \Lambda^{i-2}(F \otimes G) \otimes L_\lambda F \otimes L_\lambda G$$

(the above map is induced by the composite map

$$\Lambda^{i-1}(F \otimes G) \otimes \Lambda^t F \otimes \Lambda^t G \xrightarrow{\Delta \otimes 1} \Lambda^{i-2}(F \otimes G) \otimes F \otimes G \otimes \Lambda^t G \otimes \Lambda^t F$$
$$\xrightarrow{T} \Lambda^{i-2}(F \otimes G) \otimes \Lambda^t F \otimes F \otimes \Lambda^t G \otimes G$$
$$\xrightarrow{1 \otimes d_\lambda(F) \otimes d_\lambda(G)} \Lambda^{i-2}(F \otimes G) \otimes L_\lambda F \otimes L_\lambda G,$$

where Δ is the diagonalization, T is an appropriate twisting, and $L_\lambda(-)$ is the Schur functor corresponding the partition $\lambda = (t, 1)$). If no confusion is possible, we denote $Z_i^t(m, n)$ simply by Z_i^t.

Remark II.6.3. $\{X_i^t\}_{i>0}$ and $\{Z_i^t\}_{i>0}$ have the structure of a complex. In fact consider the commutative diagrams

$$\begin{tikzcd}[column sep=small]
 & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & X_{i+1}^t \arrow[r] & S(-i-t)\otimes\Lambda^i(F\otimes G)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-i-t)\otimes\Lambda^{i-1}(F\otimes G)S_{t+1}(F\otimes G) \arrow[d] \\
0 \arrow[r] & X_i^t \arrow[r] & S(-i-t+1)\otimes\Lambda^{i-1}(F\otimes G)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-i-t+1)\otimes\Lambda^{i-2}(F\otimes G)\otimes S_{t+1}(F\otimes G) \arrow[d] \\
 & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & X_2^t \arrow[r] & S(-1-t)\otimes\Lambda^1(F\otimes G)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-1-t)\otimes S_{t+1}(F\otimes G) \arrow[d] \\
0 \arrow[r] & X_1^t \arrow[r] & S(-t)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 & Z_{i+1}^t \arrow[r] & S(-i-t)\otimes\Lambda^i(F\otimes G)\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-i-t)\otimes\Lambda^{i-1}(F\otimes G)\otimes L_\lambda F\otimes L_\lambda G \arrow[d] \\
0 \arrow[r] & Z_i^t \arrow[r] & S(-i-t+1)\otimes\Lambda^{i-1}(F\otimes G)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-i-t+1)\otimes\Lambda^{i-2}(F\otimes G)\otimes L_\lambda F\otimes L_\lambda G \arrow[d] \\
 & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & Z_2^t \arrow[r] & S(-1-t)\otimes\Lambda^1(F\otimes G)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] \arrow[d] & S(-1-t)\otimes L_\lambda F\otimes L_\lambda G \arrow[d] \\
0 \arrow[r] & Z_1^t \arrow[r] & S(-t)\otimes\Lambda^t F\otimes\Lambda^t G \arrow[r] & 0,
\end{tikzcd}$$

where the vertical maps are induced by the Koszul map

$$\begin{aligned}
S(-i-t)\otimes\Lambda^i(F\otimes G) &\xrightarrow{1\otimes\Delta} S(-i-t)\otimes(F\otimes G)\otimes\Lambda^i(F\otimes G)\\
&\xrightarrow{1\otimes\phi\otimes 1} S(-i-t)\otimes S_1\otimes\Lambda^{i-1}(F\otimes G)\\
&\xrightarrow{m\otimes 1} S(-i-t+1)\otimes\Lambda^{i-1}(F\otimes G).
\end{aligned}$$

(ϕ is the generic map, i.e., letting $\{f_1, ..., f_m\}$ and $\{g_1, ..., g_n\}$ be free bases of F and G, respectively, $\phi: F \otimes G \to S_1$ is defined by $\phi(f_i \otimes g_j) = x_{ij}$ for all i and j; m is the multiplication of S). By the definition of X_i^t and Z_i^t, the horizontal sequences are exact. Therefore the diagrams above give the structure of graded complexes to $\mathbb{X}_\cdot^t = \{X_i^t\}_{i>0}$ and $\mathbb{Z}_\cdot^t = \{Z_i^t\}_{i>0}$.

In the rest of this section, let R be the integers $\mathbb{Z}$ or a field k. In this case both $\mathbb{X}_\cdot^t$ and $\mathbb{Z}_\cdot^t$ are graded free complexes.

PROPOSITION II.6.4 [1, Remark 3.19]. *For any m, n, and t, there exist exact sequences*

$$0 \to X_{i+1}^t \xrightarrow{\alpha_{i+1}^t} Z_{i+1}^t \xrightarrow{\beta_{i+1}^t} X_i^{t+1} \to 0 \cdots (\#)$$

for each $i \geq 0$ (we regard X_0^{t+1} as 0). Moreover these exact sequences induce the exact sequence of complexes

$$0 \to \mathbb{X}_\cdot^t \to \mathbb{Z}_\cdot^t \to \mathbb{X}_\cdot^{t+1} \to 0.$$

Sketch of Proof. By definition it is obvious that X_{i+1}^t is included in Z_{i+1}^t for any m, n, t, i. We define α_{i+1}^t to be the inclusion. By Cauchy's formula [2, Theorem III.1.4], η_i^t is decomposed as

$$\begin{aligned} & S(-i-t) \otimes \Lambda^i(F \otimes G) \otimes \Lambda^t F \otimes \Lambda^t G \\ & \xrightarrow{\xi_i^t} S(-i-t) \otimes \Lambda^{i-1}(F \otimes G) \otimes S_{t+1}(F \otimes G) \\ & \xrightarrow{1 \otimes 1 \otimes \gamma} S(-i-t) \otimes \Lambda^{i-1}(F \otimes G) \otimes \frac{S_{t+1}(F \otimes G)}{\Lambda^{t+1} F \otimes \Lambda^{t+1} G}, \end{aligned}$$

where γ is the projection. Therefore it is easy to see that $\xi_i^t(Z_{i+1}^t)$ is included in $S(-i-t) \otimes \Lambda^{i-1}(F \otimes G) \otimes \Lambda^{t+1} F \otimes \Lambda^{t+1} G$. Since $\xi_\cdot^t$ and $\eta_\cdot^t$ are induced by a Koszul map, we obtain $\xi_{i-1}^{t+1} \circ \xi_i^t(Z_{i+1}^t) = 0$. This implies that $\xi_i^t(Z_{i+1}^t)$ is included in X_i^{t+1}. β_{i+1}^t is defined to be the restriction of ξ_i^t to Z_{i+1}^t.

By using decomposition techniques similar to those in [3], one can prove the exactness of $(\#)$. It is easy to check that they induce the exact sequences of the graded complexes. Q.E.D.

Now, let us consider the graded S-free complex $\mathbb{X}_\cdot^t$. By the definition of $\mathbb{X}_\cdot^t$, X_i^t is $S(-t) \otimes \Lambda^t F \otimes \Lambda^t G$ and X_2^t is the graded S-free module generated by degree 1 relations on t-minors of (x_{ij}). Moreover by a diagram chasing argument, we obtain the exact sequences

$$0 \to (X_{i+1}^t)_{i+t} \to (X_i^t)_{i+t} \to (X_{i-1}^t)_{i+t}$$

for all $i \geq 2$. (The exact sequence above is the degree $i+t$ component of the graded complex $\mathbb{X}^t_\cdot$. Compare these exact sequences with the exact sequences in Remark II.6.1.)

Remark II.6.5. Let R be the integers $\mathbb{Z}$ or a field k. Suppose that there exists a minimal free resolution for certain integers m, n, and t. Then by the argument as above, the linear complex must be isomorphic to $\mathbb{X}^t_\cdot$ as graded S-free complex.

Therefore we call $\mathbb{X}^t_\cdot$ a linear complex, too.

III. Cauchy Formula for $S(\varphi \otimes \psi)$

In this chapter, any Hopf algebra $A = \sum_{i,j \in \mathbb{Z}} A^i_j$ is a commutative Hopf algebra in the category G^2_R which satisfies the following conditions:

(1) A^i_j is a finite free R-module for each $i, j \in \mathbb{Z}$.

(2) $A^i_j = 0$ unless $(i, j) \in \mathbb{N}^2_0$.

(3) $A^0 = R$ and u_A is the canonical inclusion. ε_A is the canonical projection, where u_A and ε_A are the unit morphism and the co-unit morphism, respectively.

Note that the tensor product of two Hopf algebras is again a Hopf algebra in the above sense. For a finite free R-module F, we let $\Lambda^i F$ (resp. $D_i F$) be degree $(i, 0)$ (resp. (i, i)) so that ΛF (resp. DF) is commutative.

1. *Pairings*

Definition III.1.1. Let A and B be Hopf algebras, and let C be a commutative algebra in the category G^2_R. For a morphism $\phi \in \mathrm{Hom}_{G^2_R}(A \otimes B, C)$ and $i \in \mathbb{N}_0$, we denote by $\phi\langle i \rangle$ the composite map

$$T_i A \otimes T_i B = A \otimes \cdots \otimes A \otimes B \otimes \cdots \otimes B \xrightarrow{T} A \otimes B \otimes \cdots \otimes A \otimes B$$

$$\xrightarrow{\phi \otimes \cdots \otimes \phi} C \otimes \cdots \otimes C \xrightarrow{m_C} C.$$

Definition III.1.2. Let A, B, and C be as above. $\phi \in \mathrm{Hom}_{G^2_R}(A \otimes B, C)$ is called *a pairing of Hopf algebras* if the following diagram is commutative:

$$\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "\mathrm{id} \otimes \Delta_B"] \arrow[d, "m_A \otimes \mathrm{id}"] & A \otimes A \otimes B \otimes B \arrow[d, "\phi\langle 2 \rangle"] & A \otimes B \otimes B \arrow[l, "\Delta_A \otimes \mathrm{id}"'] \arrow[d, "\mathrm{id} \otimes m_B"] \\
A \otimes B \arrow[r, "\phi"] & C & A \otimes B. \arrow[l, "\phi"']
\end{tikzcd}$$

LEMMA III.1.3. *Let A, B, and C be as above, and let $\phi\colon A \otimes B \to C$ be a pairing of Hopf algebras. Then the following diagram is commutative*:

$$\begin{tikzcd}
A \otimes A \otimes B \otimes B \arrow[r, "\mathrm{id} \otimes \tilde{\Box}_B"] \arrow[d, "\Box_A \otimes \mathrm{id}"] & A \otimes A \otimes B \otimes B \arrow[d, "\phi\langle 2\rangle"] & A \otimes A \otimes B \otimes B \arrow[l, "\mathrm{id} \otimes \Box_B"'] \arrow[d, "\tilde{\Box}_A \otimes \mathrm{id}"] \\
A \otimes A \otimes B \otimes B \arrow[r, "\phi\langle 2\rangle"] & C & A \otimes A \otimes B \otimes B. \arrow[l, "\phi\langle 2\rangle"']
\end{tikzcd}$$

Proof. We show the commutativity of the left rectangle. The proof for the right one is quite similar, so we omit it. For this purpose, it suffices to show that both $\phi\langle 2\rangle \circ (\mathrm{id} \otimes \tilde{\Box}_B)$ and $\phi\langle 2\rangle \circ (\Box_A \otimes \mathrm{id})$ agree with the composite map

$$A \otimes A \otimes B \otimes B \xrightarrow{\Delta_A \otimes \mathrm{id} \otimes \mathrm{id} \otimes \Delta_B} A \otimes A \otimes A \otimes B \otimes B \otimes B \xrightarrow{\phi\langle 3\rangle} C.$$

But this is clear from the commutativity of the following diagram:

$$\begin{tikzcd}
A \otimes A \otimes A \otimes B \otimes B \arrow[r, "T"] \arrow[d, "\mathrm{id} \otimes m \otimes \mathrm{id}"] \arrow[dr, phantom, "\text{(a)}"] & A \otimes B \otimes A \otimes A \otimes B \arrow[r, "\mathrm{id} \otimes \Delta"] \arrow[d, "\mathrm{id} \otimes m \otimes \mathrm{id}"] \arrow[dr, phantom, "\text{(b)}"] & A \otimes B \otimes A \otimes A \otimes B \otimes B \arrow[d, "\phi \otimes \phi\langle 2\rangle"] \\
A \otimes A \otimes B \otimes B \arrow[r, "T"] & A \otimes B \otimes A \otimes B \arrow[r, "\phi \otimes \phi"] & C \otimes C \\
A \otimes A \otimes B \otimes B \otimes B \arrow[r, "T"] \arrow[u, "\mathrm{id} \otimes m \otimes \mathrm{id}"] \arrow[ur, phantom, "\text{(c)}"] & A \otimes B \otimes B \otimes A \otimes B \arrow[r, "\Delta \otimes \mathrm{id}"] \arrow[u, "\mathrm{id} \otimes m \otimes \mathrm{id}"] \arrow[ur, phantom, "\text{(d)}"] & A \otimes A \otimes B \otimes B \otimes A \otimes B. \arrow[u, "\phi\langle 2\rangle \otimes \phi"]
\end{tikzcd}$$

The commutativities of (a) and (c) follow from the naturality of the twisting morphism T. The commutativities of (b) and (d) follow from the definition of a pairing of Hopf algebra.

DEFINITION III.1.4. Let A, B, and C be Hopf algebras, and $\phi \in \mathrm{Hom}_{G_R^2}(A \otimes B, C)$. We say that ϕ is *a Cauchy pairing* if ϕ is a homomorphism of coalgebras and a pairing of Hopf algebras, and if $\phi(A^i \otimes B^j) = 0$ for $i, j \in \mathbb{N}_0$ with $i \neq j$.

LEMMA III.1.5. *Let A, A', B, B', and C be Hopf algebras, and $\phi\colon A \otimes B \to C$ be a Cauchy pairing. If $\alpha\colon A' \to A$ and $\beta\colon B' \to B$ are homomorphisms of Hopf algebras, then the composite map $\phi \circ (\alpha \otimes \beta)$: $A' \otimes B' \to C$ is also a Cauchy pairing.*

Proof. Easy.

LEMMA III.1.6. *Let A, A', B, B', C, and C' be Hopf algebras, and*

$\phi: A\otimes B \to C$ and $\phi': A'\otimes B' \to C'$ be Cauchy pairings. Then the composite map

$$[A\otimes A']\otimes[B\otimes B'] \xrightarrow{\mathrm{id}\otimes T\otimes \mathrm{id}} A\otimes B\otimes A'\otimes B' \xrightarrow{\phi\otimes\phi'} C\otimes C'$$

is also a Cauchy pairing.

Proof. Easy.

Now let F and G be finite free R-modules. We define the morphism $\phi^S = \phi^S(F, G): \Lambda F\otimes \Lambda G \to S(F\otimes G)$ given by

$$\phi^S(f_1 \wedge \cdots \wedge f_k \otimes g_1 \wedge \cdots \wedge g_k) = (-1)^{k(k-1)/2} \det(f_i \otimes g_j)_{1\leqslant i,j\leqslant k}$$

for $k\in \mathbb{N}_0$ and $f_1, ..., f_k \in F$ and $g_1, ..., g_k \in G$, and $\phi^S(\Lambda^i F\otimes \Lambda^j G) = 0$ for $i\neq j$. We denote the restriction of ϕ^S to $\Lambda^k F\otimes \Lambda^k G$ by ϕ^S_k.

LEMMA III.1.7. *Let F and G be as above and $k\in\mathbb{N}_0$. The diagram*

$$\begin{array}{ccc} \Lambda^k F\otimes \Lambda^k G & \xrightarrow{\phi^S_k} & S_k(F\otimes G) \\ \big\downarrow{\scriptstyle \Delta\otimes\Delta} & & \big\downarrow{\scriptstyle \Delta} \\ T_k F\otimes T_k G & \xrightarrow{T} & T_k(F\otimes G) \end{array}$$

is commutative, where we are assuming that T_kF, T_kG, $T_k(F\otimes G)$, $S_k(F\otimes G)$ are of degree $(k, 0)$, $(k, 0)$, $(2k, 0)$, $(2k, 0)$, *respectively.*

Proof. Straightforward computation will show that the element $f_1 \wedge \cdots \wedge f_k \otimes g_1 \wedge \cdots \wedge g_k \in \Lambda^k F\otimes \Lambda^k G$ is mapped to

$$(-1)^{k(k-1)/2} \sum_{\sigma,\tau\in\mathfrak{S}_k} (-1)^{\sigma\tau} (f_{\sigma 1}\otimes g_{\tau 1})\otimes \cdots \otimes (f_{\sigma k}\otimes g_{\tau k}) \in T_k(F\otimes G)$$

by both $T\circ(\Delta\otimes\Delta)$ and $\Delta\circ\phi^S_k$. So the assertion is clear.

ϕ^S_k is characterized as the universal natural transformation which makes the diagram in the above lemma commutative. (In fact, if $R=\mathbb{Z}$, $\Delta_{S(F\otimes G)}: S_k(F\otimes G) \to T_k(F\otimes G)$ is injective. So such a morphism is unique.).

We now consider two actions of $\mathfrak{S}_k$ to T_kF for $k\in\mathbb{N}_0$. For $\sigma\in\mathfrak{S}_k$ and $f_1, ..., f_k \in F$, we set $\sigma(f_1\otimes \cdots \otimes f_k) = f_{\sigma^{-1}1}\otimes \cdots \otimes f_{\sigma^{-1}k}$ or $\sigma(f_1\otimes \cdots \otimes f_k) = (-1)^\sigma \cdot f_{\sigma^{-1}1}\otimes \cdots \otimes f_{\sigma^{-1}k}$. T_kF with the former action is denoted by $T^+_k F$ and T_kF with the latter action is denoted by $T^-_k F$. Hence, if $k=0$, both $T^-_k F$ and $T^+_k F$ are R with no action.

LEMMA III.1.8. *Let $k\in\mathbb{N}_0$.*

(i) $(T_k^+ F)^{\mathfrak{S}_k} = \mathrm{Im}(\Delta\colon D_k F \to T_k F)$

(ii) *If 2 is a non-zerodivisor in R, then* $(T_k^- F)^{\mathfrak{S}_k} = \mathrm{Im}(\Delta\colon \Lambda^k F \to T_k F)$.

Proof. We show (ii). If $k=0$, there is nothing to prove. We assume that $k \in \mathbb{N}$. It follows that $(T_k^- F)^{\mathfrak{S}_k} \supset \mathrm{Im}(\Delta\colon \Lambda^k F \to T_k F)$ from the cocommutativity of the comultiplication of ΛF. We have to show $(T_k^- F)^{\mathfrak{S}_k} \subset \mathrm{Im}(\Delta\colon \Lambda^k F \to T_k F)$ holds. We take an ordered basis $X = \{x_1 < \cdots < x_m\}$ $(m = \operatorname{rank} F)$ of F. $\{x_{i_1} \otimes \cdots \otimes x_{i_k} \mid 1 \leqslant i_j \leqslant m \ (1 \leqslant j \leqslant k)\}$ is an ordered basis of $T_k F$ with the lexicographic order. Let a be an element of $(T_k^- F)^{\mathfrak{S}_k}$. We show that $a \in \mathrm{Im}\, \Delta_{\Lambda F}$ by induction on the largest basis element which appears in a with a non-zero coefficient. Let $x_{i_1} \otimes \cdots \otimes x_{i_k}$ be the largest basis element and c be its coefficient in a. By the assumption on a, it holds that

$$\forall \sigma \in \mathfrak{S}_k \ x_{i_1} \otimes \cdots \otimes x_{i_k} \geqslant x_{i_{\sigma 1}} \otimes \cdots \otimes {}_{i_{\sigma k}}.$$

So $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_k$. But if the equality occurs somewhere, say $i_j = i_{j+1}$, then $c = -c$, since $a = (i_j i_{j+1}) \cdot a$ by assumption. Since 2 is a non-zerodivisor in R, it is a contradiction. So we have $i_1 > i_2 > \cdots > i_k$. Now consider the element $a' = a - c \cdot \Delta_{\Lambda F}(x_{i_1} \wedge \cdots \wedge x_{i_k})$. a' is an element of $(T_k^- F)^{\mathfrak{S}_k}$, and the largest basis element which appears in a' with non-zero coefficient is smaller than $x_{i_1} \otimes \cdots \otimes x_{i_k}$. So $a' \in \mathrm{Im}\, \Delta_{\Lambda F}$ by assumption of induction. This shows that $a \in \mathrm{Im}\, \Delta_{\Lambda F}$. The proof for (i) is quite similar, so we omit it. Q.E.D.

LEMMA III.1.9. *Let F and G be finite R-modules, and* $k \in \mathbb{N}_0$. *Then there exist universal natural transformations* $\phi_k^\Lambda = \phi_k^\Lambda(F, G)$, $\psi_k^\Lambda = \psi_k^\Lambda(F, G)$, *and* $\psi_k^D = \psi_k^D(F, G)$ *which make the diagrams*

$$\begin{tikzcd}
D_k F \otimes \Lambda^k G \arrow[r, "\phi_k^\Lambda"] \arrow[d, "\Delta \times \Delta"] \arrow[dr, phantom, "\text{(a)}"] & \Lambda^k(F \otimes G) \arrow[d, "\Delta"] & \Lambda^k F \otimes D_k G \arrow[r, "\psi_k^\Lambda"] \arrow[d, "\Delta \times \Delta"] \arrow[dr, phantom, "\text{(b)}"] & \Lambda^k(F \otimes G) \arrow[d, "\Delta"] & D_k F \otimes D_k G \arrow[r, "\psi_k^D"] \arrow[d, "\Delta \times \Delta"] \arrow[dr, phantom, "\text{(c)}"] & D_k(F \otimes G) \arrow[d, "\Delta"] \\
T_k^+ F \otimes T_k^- G \arrow[r, "T"] & T_k^-(F \otimes G) & T_k^- F \otimes T_k^+ G \arrow[r, "T"] & T_k^-(F \otimes G) & T_k^+ F \otimes T_k^+ G \arrow[r, "T"] & T_k^+(F \otimes G),
\end{tikzcd}$$

commutative uniquely, where we are assuming that $T_k^+ F$ *and* $T_k^+ G$ *are of degree* (k, k), $T_k^- F$ *and* $T_k^- G$ *are of degree* $(k, 0)$, $\Lambda^k(F \otimes G)$ *and* $T_k^-(F \otimes G)$ *are of degree* $(2k, k)$, *and* $D_k(F \otimes G)$ *and* $T_k^+(F \otimes G)$ *are of degree* $(2k, 2k)$.

Proof. The bottom arrow T of each diagram is a homomorphism of $\mathfrak{S}_k$-module. So the assertion follows from (III.1.8) easily. Q.E.D.

We define $\phi^\Lambda = \phi^\Lambda(F, G)\colon DF \otimes \Lambda G \to \Lambda(F \otimes G)$ as the map given by $\phi^\Lambda = \phi_k^\Lambda$ on $D_k F \otimes \Lambda^k G$ and $\phi^\Lambda(D_i F \otimes \Lambda^j G) = 0$ for $i \neq j$. $\psi^\Lambda = \psi^\Lambda(F, G)$:

$\Lambda F \otimes DG \to \Lambda(F \otimes G)$ and $\psi^D = \psi^D(F, G)\colon DF \otimes DG \to D(F \otimes G)$ are defined in a similar way.

Let us set

$$\mathfrak{S}_{i,j} = \{\sigma \in \mathfrak{S}_{i+j} \mid \sigma 1 < \cdots < \sigma i, \sigma(i+1) < \cdots < \sigma(i+j)\}$$

for $i, j \in \mathbb{N}_0$. Note that the comultiplications of SF and ΛF are given by

$$\begin{aligned} &\Delta_{SF}(f_1 \cdots f_k) \\ &\quad = \sum_{i+j=k} \sum_{\sigma \in \mathfrak{S}_{i,j}} f_{\sigma 1} \cdots f_{\sigma i} \otimes f_{\sigma(i+1)} \cdots f_{\sigma k} \\ &\Delta_{\Lambda F}(f_1 \wedge \cdots \wedge f_k) \\ &\quad = \sum_{i+j=k} \sum_{\sigma \in \mathfrak{S}_{i,j}} (-1)^{\sigma} \cdot f_{\sigma 1} \wedge \cdots \wedge f_{\sigma i} \otimes f_{\sigma(i+1)} \wedge \cdots \wedge f_{\sigma k}, \end{aligned}$$

where $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in F$.

LEMMA III.1.10. *Let F be a finite free R-module and $i, j \in \mathbb{N}_0$. The following diagrams are commutative:*

$$\begin{tikzcd} S_iF \otimes S_jF \arrow[r, "m_{SF}"] \arrow[d] & S_{i+j}F \arrow[d] \\ T_i^+F \otimes T_j^+F \arrow[r, "\sum_{\sigma \in \mathfrak{S}_{i,j}} \sigma"] & T_{i+j}^+F \end{tikzcd} \qquad \begin{tikzcd} \Lambda^iF \otimes \Lambda^jF \arrow[r, "m_{\Lambda F}"] \arrow[d] & \Lambda^{i+j}F \arrow[d] \\ T_i^-F \otimes T_j^-F \arrow[r, "\sum_{\sigma \in \mathfrak{S}_{i,j}} \sigma"] & T_{i+j}^-F \end{tikzcd}$$

$$\begin{tikzcd} D_iF \otimes D_jF \arrow[r, "m_{DF}"] \arrow[d] & D_{i+j}F \arrow[d] \\ T_i^+F \otimes T_j^+F \arrow[r, "\sum_{\sigma \in \mathfrak{S}_{i,j}} \sigma"] & T_{i+j}^+F, \end{tikzcd}$$

where the vertical arrows of each diagram are diagonalizations, and all tensor functors are graded so as to all arrows are homogeneous. We assume that $\sum_{\sigma \in \mathfrak{S}_{i,j}} \sigma$ is the identity map if $i = j = 0$.

Proof. Straightforward computation using the above remark shows the commutativity of the last two diagrams. The commutativity of the first diagram is clear, since it consists of universally free functors and universal morphisms, and it agrees with the first diagram (up to grading) if $R = \mathbb{Q}$.

PROPOSITION III.1.11. *ϕ^S, ϕ^Λ, ψ^Λ, and ψ^D are Cauchy pairings.*

Proof. We show only that ϕ^S is a Cauchy pairing. The proofs for the others are quite similar.

First, we want to show that ϕ^S is a homomorphism of coalgebras. For this purpose, it is sufficient to show that the following diagram is commutative for all $i, j \in \mathbb{N}_0$.

$$\begin{tikzcd}
\Lambda^{i+j}F\otimes\Lambda^{i+j}G \arrow[r, "\phi^S_{i+j}"] \arrow[d, "\Delta_{\Lambda F\otimes\Lambda G}"] & S_{i+j}(F\otimes G) \arrow[d, "\Delta_{S(F\otimes G)}"'] \\
\Lambda^iF\otimes\Lambda^iG\otimes\Lambda^jF\otimes\Lambda^jG \arrow[r, "\phi^S_i\otimes\phi^S_j"] & S_i(F\otimes G)\otimes S_j(F\otimes G)
\end{tikzcd} \tag{$*$}$$

Since the above diagram $(*)$ is universal, we may assume that $R=\mathbb{Z}$. From (III.1.7), the whole rectangle and the lower rectangle of the following diagram are commutative:

$$\begin{tikzcd}
\Lambda^{i+j}F\otimes\Lambda^{i+j}G \arrow[r, "\phi^S_{i+j}"] \arrow[d, "\Delta_{\Lambda F\otimes\Lambda G}"] & S_{i+j}(F\otimes G) \arrow[d, "\Delta_{S(F\otimes G)}"] \\
\Lambda^iF\otimes\Lambda^iG\otimes\Lambda^jF\otimes\Lambda^jG \arrow[r, "\phi^S_i\otimes\phi^S_j"] \arrow[d, "\Delta\otimes\Delta\otimes\Delta\otimes\Delta"] & S_i(F\otimes G)\otimes S_j(F\otimes G) \arrow[d, "\Delta\otimes\Delta"] \\
T_iF\otimes T_iG\otimes T_jF\otimes T_jG \arrow[r, "T\otimes T"] & T_i(F\otimes G)\otimes T_j(F\otimes G)
\end{tikzcd}$$

Since $\Delta\otimes\Delta : S_i(F\otimes G)\otimes S_j(F\otimes G) \to T_i(F\otimes G)\otimes T_j(F\otimes G)$ is injective, the diagram $(*)$ is commutative. Hence, ϕ^S is a homomorphism of coalgebra.

Next, we show that ϕ^S is a pairing of Hopf algebra. We may assume that $R=\mathbb{Z}$. Consider the diagram

$$\begin{tikzcd}
\Lambda^{i+j}F\otimes\Lambda^{i+j}G \arrow[r] & T^-_{i+j}F\otimes T^-_{i+j}G \arrow[dddd, bend left=60, "T"] \\
\Lambda^iF\otimes\Lambda^jF\otimes\Lambda^{i+j}G \arrow[u, "m\otimes\mathrm{id}"] \arrow[r] \arrow[d, "\mathrm{id}\otimes\Delta"] \arrow[ur, phantom, "\text{(a)}"] & T^-_iF\otimes T^-_jF\otimes T^-_{i+j}G \arrow[u, "\sum_{\sigma\in\mathfrak{S}_{i,j}}\sigma\otimes\sigma"'] \arrow[d, "\simeq"'] \\
\Lambda^iF\otimes\Lambda^jF\otimes\Lambda^iG\otimes\Lambda^jG \arrow[r] \arrow[d, "{[\phi^S_i\otimes\phi^S_j]\circ T}"] \arrow[ur, phantom, "\text{(b)}"] & T^-_iF\otimes T^-_jF\otimes T^-_iG\otimes T^-_jG \arrow[d, "T\quad\text{(e)}"'] \\
S_i(F\otimes G)\otimes S_j(F\otimes G) \arrow[r] \arrow[d, "m"] \arrow[ur, phantom, "\text{(c)}"] & T^+_i(F\otimes G)\otimes T^+_j(F\otimes G) \arrow[d, "\sum_{\sigma\in\mathfrak{S}_{i,j}}\sigma"'] \\
S_{i+j}(F\otimes G) \arrow[r] \arrow[ur, phantom, "\text{(d)}"] & T^+_{i+j}(F\otimes G)
\end{tikzcd}$$

where $i, j \in \mathbb{N}_0$. Since $R=\mathbb{Z}$, $\Lambda^{i+j}G=(T^-_{i+j}G)^{\mathfrak{S}_{i+j}}$. From this fact and

(III.1.10), (a) and (d) commute. (b) commutes from the coassociativity of $\Delta_{\Lambda G}$. (c) is commutative, since ϕ^S is a homomorphism of coalgebra. (e) is also commutative, since $T\colon T^-_{i+j}F\otimes T^-_{i+j}G \to T^+_{i+j}(F\otimes G)$ is a homomorphism of $\mathfrak{S}_{i+j}$-module.

So it is easy to check that (f) of the following diagram is commutative:

$$\begin{tikzcd}
\Lambda^iF\otimes\Lambda^jF\otimes\Lambda^{i+j}G \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "m\otimes\mathrm{id}"] \arrow[dr, phantom, "\text{(f)}"] & \Lambda^iF\otimes\Lambda^jF\otimes\Lambda^iG\otimes\Lambda^jG \arrow[d, "\phi^S\langle 2\rangle"] \arrow[dr, phantom, "\text{(g)}"] & \Lambda^{i+j}F\otimes\Lambda^iG\otimes\Lambda^jG \arrow[l, "\Delta\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes m"] \\
\Lambda^{i+j}F\otimes\Lambda^{i+j}G \arrow[r, "\phi^S"] & S_{i+j}(F\otimes G) & \Lambda^{i+j}F\otimes\Lambda^{i+j}G \arrow[l, "\phi^S"']
\end{tikzcd}$$

The commutativity of (g) is shown similarly. Since (f) and (g) are commutative, ϕ^S is a pairing of Hopf algebras. Hence, ϕ^S is a Cauchy pairing. Q.E.D.

2. *Cauchy Formula for* $S(\varphi\otimes\psi)$

Let $\varphi\colon F_1\to F_0$ and $\psi\colon G_1\to G_0$ be morphisms of finite free R-module. $\Lambda\varphi$, $\Lambda\psi$, and $S(\varphi\otimes\psi)$ are graded in the manner mentioned in Sect. 2 of Chap. I.

Definition III.2.1. We define a natural transformation $\theta=\theta(\varphi,\psi)$: $\Lambda\varphi\otimes\Lambda\psi\to S(\varphi\otimes\psi)$ as the composite morphism

$$\begin{aligned}
&\Lambda\varphi\otimes\Lambda\psi\\
&\xrightarrow{\Delta\otimes\Delta} \Lambda\varphi\otimes\Lambda\varphi\otimes\Lambda\psi\otimes\Lambda\psi = \Lambda F_0\otimes DF_1\otimes\Lambda F_0\otimes DF_1\otimes\Lambda G_0\otimes DG_1\otimes\Lambda G_0\otimes DG_1\\
&\xrightarrow{T} \Lambda F_0\otimes\Lambda G_0\otimes DF_1\otimes\Lambda G_0\otimes\Lambda F_0\otimes DG_1\otimes DF_1\otimes DG_1\\
&\xrightarrow{\phi^S\otimes\phi^\Lambda\otimes\psi^\Lambda\otimes\psi^D} S(F_0\otimes G_0)\otimes\Lambda(F_1\otimes G_0)\otimes\Lambda(F_0\otimes G_1)\otimes D(F_1\otimes G_1)\\
&\xrightarrow{\simeq} S(F_0\otimes G_0)\otimes\Lambda(F_1\otimes G_0\oplus F_0\otimes G_1)\otimes D(F_1\otimes G_1)=S(\varphi\otimes\psi).
\end{aligned}$$

From (III.1.5), (III.1.6), and (III.1.11), θ is a Cauchy pairing, since $\Delta_{\Lambda\varphi}$, $\Delta_{\Lambda\psi}$, and T are homomorphisms of Hopf algebras. Clearly, θ is universal. Furthermore, we have

Lemma III.2.2. *θ is a chain map.*

Proof. We have to show that $\partial_{S(\varphi\otimes\psi)}\circ\theta=\theta\circ\partial_{\Lambda\varphi\otimes\Lambda\psi}$. Since all of the morphisms we are considering are universal, we may assume that $R=\mathbb{Z}$. Since θ is a homomorphism of coalgebras, the following diagram is commutative for $k\in\mathbb{N}_0$:

$$\begin{tikzcd}
\Lambda^k\varphi\otimes\Lambda^k\psi \arrow[r, "\theta_k"] \arrow[dr, "\Delta_{\Lambda\varphi\otimes\Lambda\psi}"'] & S_k(\varphi\otimes\psi) \arrow[d, "\Delta_{S(\varphi\otimes\psi)}"] \\
& T_k(\varphi\otimes\psi)
\end{tikzcd}$$

Here θ_k is the restriction of θ to $\Lambda^k\varphi \otimes \Lambda^k\psi$. Since $\Delta_{S(\varphi\otimes\psi)}$ and $\Delta_{\Lambda\varphi\otimes\Lambda\psi}$ are chain maps, and $\Delta_{S(\varphi\otimes\psi)}$ is injective, θ_k is a chain map. Hence, θ is also a chain map.

Let $\lambda \in \Omega_k^+$. We define $\theta_\lambda\colon \Lambda_\lambda\varphi \otimes \Lambda_\lambda\psi \to S_k(\varphi\otimes\psi)$ to be the restriction of $\theta\langle q\rangle\colon T_q(\Lambda\varphi)\otimes T_q(\Lambda\psi) \to S_k(\varphi\otimes\psi)$, where $q = \lg(\lambda)$. We also define ϕ_λ^S, ϕ_λ^Λ, ψ_λ^Λ, and ψ_λ^D in a similar way.

DEFINITION III.2.3. Let $k \in \mathbb{N}_0$ and $\lambda \in \Omega_k^-$. We define $M^\lambda(\theta) = M^\lambda(\theta)(\varphi, \psi)$ and $\dot{M}^\lambda(\theta) = \dot{M}^\lambda(\theta)(\varphi, \psi)$ as subcomplexes of $S_k(\varphi\otimes\psi)$ given by

$$M^\lambda(\theta) = \sum_{\mu\in\Omega_k^-,\, \mu \geqslant \lambda} \operatorname{Im}\theta_\mu \qquad \text{and} \qquad \dot{M}^\lambda(\theta) = \sum_{\mu\in\Omega_k^-,\, \mu > \lambda} \operatorname{Im}\theta_\mu.$$

We also define $M^\lambda(\phi^S)$, $\dot{M}^\lambda(\phi^S)$, $M^\lambda(\phi^\Lambda)$, $\dot{M}^\lambda(\phi^\Lambda)$, $M^\lambda(\psi^\Lambda)$, $\dot{M}^\lambda(\psi^\Lambda)$, $M^\lambda(\psi^D)$ and $\dot{M}^\lambda(\psi^D)$ in a similar way.

LEMMA III.2.4. *Let $k \in \mathbb{N}_0$ and $\lambda \in \Omega_k^+$ with $\lg(\lambda) = q$. If $\sigma \in \mathfrak{S}_q$, the following diagram is commutative:*

$$\begin{tikzcd}
\Lambda^\lambda\varphi \otimes \Lambda^\lambda\psi \arrow[r, "T_\sigma^{-1}"] \arrow[dr, "\theta_\lambda"'] & \Lambda^{\sigma\cdot\lambda}\varphi \otimes \Lambda^{\sigma\cdot\lambda}\psi \arrow[d, "\theta_{\sigma\cdot\lambda}"] \\
 & S_k(\varphi\otimes\psi)
\end{tikzcd}$$

Here $\sigma\cdot\lambda$ is the sequence $(\lambda_{\sigma 1}, ..., \lambda_{\sigma q})$ and $T_{\sigma^{-1}}$ is the restriction of the action of $\sigma^{-1}\otimes\sigma^{-1}$ to $T_q(\Lambda\varphi)\otimes T_q(\Lambda\psi)$ via twisting.

Proof. Trivial from the naturality of T and the commutativity of the multiplication of $S(\varphi\otimes\psi)$.

LEMMA III.2.5. *Let $k \in \mathbb{N}_0$. It holds that $M^{\omega_k}(\theta) = S_k(\varphi\otimes\psi)$. Hence, $S_k(\varphi\otimes\psi)$ admits a filtration $\{M^\lambda(\theta)\}_{\lambda\in\Omega_k^-}$.*

Proof. By definition, $S_k(\varphi\otimes\psi)$ is the direct sum

$$\sum_{\alpha+\beta+\gamma+\delta=k} S_\alpha(F_0\otimes G_0)\otimes\Lambda_\beta(F_1\otimes G_0)\otimes\Lambda_\gamma(F_0\otimes G_1)\otimes D_\delta(F_1\otimes G_1).$$

So it suffices to prove that $S_\alpha(F_0\otimes G_0)\otimes\Lambda_\beta(F_1\otimes G_0)\otimes\Lambda_\gamma(F_0\otimes G_1)\otimes D_\delta(F_1\otimes G_1)$ is contained in $M^{\omega_k}(\theta)$ for each α, β, γ and δ. Since the multiplications of $S_\alpha(F_0\otimes G_0)$, $\Lambda_\beta(F_1\otimes G_0)$, and $\Lambda_\gamma(F_0\otimes G_1)$ are surjective, it is sufficient to prove:

$$D_\delta(F_1\otimes G_1) = \sum_{\mu\in\Omega_\delta^-} \theta_\mu(D_\mu F_1\otimes D_\mu G_1) = M^{\omega_\delta}(\psi^D)(F_1, G_1). \qquad (*)$$

In fact, if $(*)$ is true, $S_\alpha(F_0\otimes G_0)\otimes\Lambda_\beta(F_1\otimes G_0)\otimes\Lambda_\gamma(F_0\otimes G_1)\otimes D_\delta(F_1\otimes G_1)$ is contained in $\sum_{\lambda\in\Omega_k^-,\lg(\lambda)>\alpha+\beta+\gamma}\operatorname{Im}\theta_\lambda$.

We show $(*)$. Let $X=\{x_1,...,x_m\}$ (resp. $Y=\{y_1,...,y_n\}$) be a basis of F_1 (resp. G_1). A basis element of $D_\delta(F_1\otimes G_1)$ is written as $\prod_{i=1}^m\prod_{j=1}^n(x_i\otimes y_j)^{(v_{i,j})}$, where $v=(v_{1,1},...,v_{1,n},...,v_{m,1},...,v_{m,n})\in\Omega_\delta^+$ and $(x_i\otimes y_j)^{(v_{i,j})}$ is the $v_{i,j}$th divided power of $x_i\otimes y_j$ (see, for example, [2]). It is easy to check that $\psi_r^D(x_i^{(r)}\otimes y_j^{(r)})=(x_i\otimes y_j)^{(r)}$ for any $r\in\mathbb{N}_0$. So this basis element is contained in $\operatorname{Im}\psi_v(F_1,G_1)$. From (III.2.4), $\operatorname{Im}\psi_v(F_1,G_1)$ $=\theta_v(D_vF_1\otimes D_vG_1)=\theta_{\tilde{v}}(D_{\tilde{v}}F_1\otimes D_{\tilde{v}}G_1)\subset\sum_{\mu\in\Omega_\delta^-}\theta_\mu(D_\mu F_1\otimes D_\mu G_1)$. Therefore, $(*)$ is true and we have completed the proof. Q.E.D.

PROPOSITION III.2.6. *Let $\lambda\in\Omega_k^-$ and $\mu\in S_\square(\lambda)$. The following diagram is commutative:*

$$\begin{tikzcd}
\Lambda_\mu\varphi\otimes\Lambda_\lambda\psi \arrow[r, "\mathrm{id}\otimes\dot{\square}_\mu(\Lambda\psi)"] \arrow[d, "(\Lambda\varphi)\otimes\mathrm{id}"] & \Lambda_\mu\varphi\otimes\Lambda_\mu\psi \arrow[d, "\theta_\mu"] & \Lambda_\lambda\varphi\otimes\Lambda_\mu\psi \arrow[l, "\dot{\square}_\mu(\Lambda\varphi)\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\square_\lambda(\Lambda\psi)"] \\
\Lambda_\lambda\varphi\otimes\Lambda_\lambda\psi \arrow[r, "\theta_\lambda"] & S_k(\varphi\otimes\psi) & \Lambda_\lambda\varphi\otimes\Lambda_\lambda\psi \arrow[l, "\theta_\lambda"']
\end{tikzcd}$$

Proof. We put $\mu=\lambda+r\cdot\alpha_t$ ($t,r\in\mathbb{N}$, $t<\lg(\lambda)$, $r\leqslant\lambda_{t+1}$). From (III.1.3), the diagram

$$\begin{tikzcd}
\Lambda^{\lambda_t+r}\varphi\otimes\Lambda^{\lambda_{t+1}-r}\varphi\otimes\Lambda^{\lambda_t}\psi\otimes\Lambda^{\lambda_{t+1}}\psi \arrow[r, "\square(\Lambda\varphi)\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\dot{\square}(\Lambda\psi)"] & \Lambda^{\lambda_t}\varphi\otimes\Lambda^{\lambda_{t+1}}\varphi\otimes\Lambda^{\lambda_t}\psi\otimes\Lambda^{\lambda_{t+1}}\psi \arrow[d, "\theta\langle 2\rangle"] \\
\Lambda^{\lambda_t+r}\varphi\otimes\Lambda^{\lambda_{t+1}-r}\varphi\otimes\Lambda^{\lambda_t+r}\psi\otimes\Lambda^{\lambda_{t+1}-r}\psi \arrow[r, "\theta\langle 2\rangle"] & S_{\lambda_t+\lambda_{t+1}}(\varphi\otimes\psi) \\
\Lambda^{\lambda_t}\varphi\otimes\Lambda^{\lambda_{t+1}}\varphi\otimes\Lambda^{\lambda_t+r}\psi\otimes\Lambda^{\lambda_{t+1}-r}\psi \arrow[r, "id\otimes\square(\Lambda\psi)"] \arrow[u, "\dot{\square}(\Lambda\varphi)\otimes\mathrm{id}"] & \Lambda^{\lambda_t}\varphi\otimes\Lambda^{\lambda_{t+1}}\varphi\otimes\Lambda^{\lambda_t}\psi\otimes\Lambda^{\lambda_{t+1}}\psi \arrow[u, "\theta\langle 2\rangle"']
\end{tikzcd}$$

is commutative, since $\theta(\Lambda^i\varphi\otimes\Lambda^j\psi)=0$ for $i,j\in\mathbb{N}_0$ with $i\neq j$. The proposition follows from the commutativity of this diagram.

We are now ready to state the Cauchy formula for $S(\varphi\otimes\psi)$.

THEOREM III.2.7 (Cauchy formula for $S(\varphi\otimes\psi)$). *Let $k\in\mathbb{N}_0$, and $\varphi:F_1\to F_0$ and $\psi:G_1\to G_0$ be morphisms of finite free R-modules. If $\lambda\in\Omega_k^-$, θ_λ induces the isomorphism of complex $\beta_\lambda:L_\lambda\varphi\otimes L_\lambda\psi\to M^\lambda(\theta)/\dot{M}^\lambda(\theta)$ which makes the following diagram commutative:*

$$\begin{tikzcd}
\Lambda_\lambda\varphi\otimes\Lambda_\lambda\psi \arrow[r, "\theta_\lambda"] \arrow[d, "d_\lambda\otimes d_\lambda"] & M^\lambda(\theta) \arrow[d, "\mathrm{proj.}"] \\
L_\lambda\varphi\otimes L_\lambda\psi \arrow[r, "\beta_\lambda"] & M^\lambda(\theta)/\dot{M}^\lambda(\theta).
\end{tikzcd}$$

Hence, the associated graded complex of the filtration $\{M^{\lambda}(\theta)\}_{\lambda \in \Omega_k^-}$ *is* $\sum_{\lambda \in \Omega_k^-} L_\lambda \varphi \otimes L_\lambda \psi$.

Proof. From (I.3.11), the kernel of the map $d_\lambda(\varphi) \otimes d_\lambda(\psi)$ is $\operatorname{Im} \square_\lambda(\Lambda\varphi) \otimes \Lambda_\lambda \psi + \Lambda_\lambda \varphi \otimes \operatorname{Im} \square_\lambda(\Lambda\psi)$. From (III.2.6), we have

$$\theta_\lambda(\operatorname{Im} \square_\lambda(\Lambda\varphi) \otimes \Lambda_\lambda \psi + \Lambda_\lambda \varphi \otimes \operatorname{Im} \square_\lambda(\Lambda\psi))$$
$$\subset \sum_{\mu \in S_\square(\lambda)}{}' \theta_\mu(\Lambda_\mu \varphi \otimes \operatorname{Im} \tilde{\square}_\mu^\lambda(\Lambda\psi) + \operatorname{Im} \tilde{\square}_\mu^\lambda(\Lambda\varphi) \otimes \Lambda_\mu \psi) \subset \dot{M}^\lambda(\theta),$$

since $\operatorname{Im} \theta_\mu = \operatorname{Im} \theta_{\tilde{\mu}}$ from (III.2.4), and $\tilde{\mu} > \lambda$ for $\mu \in S_\square(\lambda)$. Hence, θ_λ induces β_λ. By definitions of $M^\lambda(\theta)$ and $\dot{M}^\lambda(\theta)$, $\text{proj.} \circ \theta_\lambda$ is surjective. So β_λ is also surjective. We know

$$\operatorname{rank} S_k(\varphi \otimes \psi) = \operatorname{rank}\left[\sum_{\lambda \in \Omega_k^-} L_\lambda \varphi \otimes L_\lambda \psi\right]$$

from (I.4.11). So β_λ must be an isomorphism for each λ. The last assertion is now clear.

Remark III.2.8. We have the Cauchy formula for $S(F \otimes G)$ [2, Theorem III.1.4], by letting $\varphi = 0 \to F$ and $\psi = 0 \to G$ in the theorem. We also have the Cauchy formula for $\Lambda(F \otimes G)$ [2, Theorem III.2.4], by letting $\varphi = 0 \to F$ and $\psi = G \to 0$ in the theorem. In fact, ϕ_K^S and ψ_k^Λ agree with the pairings which appear in Chapter III of [2] up to sign.

COROLLARY III.2.9 (Cauchy formula for $D(F \otimes G)$). *Let* $k \in \mathbb{N}_0$, *and* F *and* G *be finite free R-modules. Then for* $\lambda \in \Omega_k^-$, ψ_λ^D *induces the isomorphism* $\beta_\lambda^D : K_\lambda F \otimes K_\lambda G \to M^\lambda(\psi^D)/\dot{M}^\lambda(\psi^D)$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} D_\lambda F \otimes D_\lambda G & \xrightarrow{\psi_\lambda^D} & M^\lambda(\psi^D) \\ \downarrow{\scriptstyle d'_\lambda \otimes d'_\lambda} & & \downarrow{\scriptstyle \text{proj.}} \\ K_\lambda F \otimes K_\lambda G & \xrightarrow{\beta_\lambda^D} & M^\lambda(\psi^D)/\dot{M}^\lambda(\psi^D). \end{array}$$

Hence, the associated graded module of the filtration $\{M^\lambda(\psi^D)\}_{\lambda \in \Omega_k^-}$ *is* $\sum_{\lambda \in \Omega_k^-} K_\lambda F \otimes K_\lambda G$.

Proof. Let $\varphi = F \to 0$ and $\psi = G \to 0$ in the theorem θ is merely ψ^D in this case. So the corollary is clear.

IV. THE LOWER SYZYGIES OF DETERMINANTAL IDEALS

Throughout this chapter, R is a commutative ring with unit F and G are free R modules of rank m and n, respectively. S, S_i, Λ, and Λ^i stand for $S(F\otimes G)$, $S_i(F\otimes G)$, $\Lambda(F\otimes G)$, and $\Lambda^i(F\otimes G)$, respectively. So $S=S(F\otimes G)$ is a graded polynomial ring in $m\cdot n$ variables over R with each variable being of degree one as in Chapter II. We denote by $I_t(\subset S)$ the determinantal ideal for any t with $1\leqslant t\leqslant \min(m,n)$.

The main purpose of this chapter is to calculate the third and fourth Betti numbers of S/I_t in the case $m=n=t+2$. More generally, we show that the third Betti numbers do not depend on the characteristic of the ground field, if $t=\min(m,n)-2$. For the fourth Betti numbers, we need more calculation. The complete proof of the independence of the characteristic will be given in Chapter V.

1. *The Filtration of Koszul Complex $\mathscr{J}^{t,r}$*

It is well-known that the Koszul complex $S(\mathrm{id}_{F\otimes G})=S\otimes\Lambda$ is the minimal free resolution of S/I_1. Hence, we have

$$\underline{\mathrm{Tor}}_i^S(S/I_t, S/I_1)\simeq \underline{H}_{i-1}(I_t\otimes_S S(\mathrm{id}_{F\otimes G}))$$

as graded S-module for $i\in\mathbb{N}$, where $\underline{H}_*$ is the homology with the grading.

DEFINITION IV.1.1. We denote the graded S-complex $I_t\otimes_S S(\mathrm{id}_{F\otimes G})$ by $\mathscr{J}^t$. We denote the degree r part of $\mathscr{J}^t$ (in the usual S-grading) by $\mathscr{J}^{t,r}$ for $r\in\mathbb{N}_0$. Hence $\mathscr{J}^{t,r}$ is an R-complex given by

$$\mathscr{J}^{t,r}=(0\to I_{t,t}\otimes\Lambda^{r-t}\to I_{t,t+1}\otimes\Lambda^{r-t-1}\to\cdots\to I_{t,r}\to 0),$$

where $I_{t,i}$ is the degree i part of I_t. We denote the graded S-complex $I_t\otimes_S S(\mathrm{id}_F\otimes\mathrm{id}_G)$ by $\tilde{\mathscr{J}}^t$. The degree r part of $\tilde{\mathscr{J}}^t$ is denoted by $\tilde{\mathscr{J}}^{t,r}$.

By definition, $\mathscr{J}^{t,r}$ (resp. $\tilde{\mathscr{J}}^{t,r}$) is a subcomplex of $S_r(\mathrm{id}_{F\otimes G})$ (resp. $S_r(\mathrm{id}_F\otimes\mathrm{id}_G)$). Since $\mathscr{J}^t=I_t\otimes\Lambda$ and $\tilde{\mathscr{J}}^t=I_t\otimes\Lambda\otimes\Lambda(F\otimes G)\otimes D(F\otimes G)$, we may assume that $\mathscr{J}^t\subset\tilde{\mathscr{J}}^t$ as an S-complex.

We denote the partition $(t,\underbrace{1,1,\ldots,1}_{r-t})\in\Omega_r^-$ by $\omega_{t,r}$ for $t\in\mathbb{N}$ and $r\geqslant t$.

LEMMA IV.1.2. *Let $r\in\mathbb{N}_0$. $\mathscr{J}^{t,r}$ and $\tilde{\mathscr{J}}^{t,r}$ are universally free R-complexes (i.e., R-complexes consisting of universally free modules and universal boundary maps).*

Proof. It is sufficient to show that $I_{t,r}$ is universally free and the canonical injection $I_{t,r}\hookrightarrow S_r(F\otimes G)$ is universal. But this is easy from (II.1.2).

Remark IV.1.3. We may assume that $\mathscr{J}^t$ and $\tilde{\mathscr{J}}^t$ are objects of $\mathscr{C}$ (see Chapter I) so that $\mathscr{J}^t$ and $\tilde{\mathscr{J}}^t$ are subobjects of $S(\mathrm{id}_{F\otimes G})$ and $S(\mathrm{id}_F\otimes\mathrm{id}_G)$, respectively. Note that $\mathscr{J}^{t,r}$ and $\tilde{\mathscr{J}}^{t,r}$ are of degree $2\cdot r$ part of $\mathscr{J}^t$ and $\tilde{\mathscr{J}}^t$, respectively (do not confuse the usual S-grading and the grading defined in chapter I). Moreover, we can make $\tilde{\mathscr{J}}^t$ and $S(\mathrm{id}_F\otimes\mathrm{id}_G)$ into double complexes in the category G_R. More generally, we can make $S(\varphi \otimes \psi)$ into a double complex for any morphisms of finite free R-modules $\varphi: F_1 \to F_0$ and $\psi: G_1 \to G_0$. We let $S_\alpha(F_0 \otimes G_0) \otimes \Lambda^\beta(F_1 \otimes G_0) \otimes \Lambda^\gamma(F_0 \otimes G_1) \otimes D_\delta(F_1 \otimes G_1)$ be of degree $(2(\alpha+\beta+\gamma+\delta),\ \beta+\delta,\ \gamma+\delta)$, where the first component indicates the degree as an object in G_R, and the second and the third components indicate the double grading as an R-double complex. (In fact, the boundary map ∂ is a sum $\partial' \pm \partial''$, where ∂' and ∂'' are maps of degree $(0, -1, 0)$ and $(0, 0, -1)$, respectively. Hence, this grading makes $(S(\varphi \otimes \psi), \partial', \partial'')$ into a double complex so that $\mathrm{Tot}(S(\varphi \otimes \psi), \partial', \partial'') = (S(\varphi \otimes \psi), \partial)$.) It is clear that $\tilde{\mathscr{J}}^t$ is converted into a double complex so as to be a subobject of $S(\mathrm{id}_F \otimes \mathrm{id}_G)$. For $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0$ with $k = \alpha + \beta$, we let $\Lambda^\alpha F_0 \otimes D_\beta F_1$ be of degree $(k, \beta, 0)$ and $\Lambda^\alpha G_0 \otimes D_\beta G_1$ be of degree $(k, 0, \beta)$ so that $\Lambda^k \varphi$ and $\Lambda^k \psi$ are double complexes. Since $\Lambda_\lambda \varphi$ and $\Lambda_\lambda \psi$ are tensor products of the double complexes graded as above, we can make them into double complexes with total gradings. Note that $\theta_\lambda: \Lambda_\lambda \varphi \otimes \Lambda_\lambda \psi \to S_{|\lambda|}(\varphi \otimes \psi)$ is homogeneous, for $\lambda \in \Omega^+$.

LEMMA IV.1.4. *For $r \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$, we have isomorphisms*

$$[\underline{\mathrm{Tor}}^S_{i+1}(S/I_t, S/I_1)]_r \simeq H_i(\mathscr{J}^{t,r}) \simeq H_i(\tilde{\mathscr{J}}^{t,r}).$$

Proof. The first isomorphism is clear. We show the second one. Since $\tilde{\mathscr{J}}^{t,r}$ is a double complex with the grading explained above, we have a spectral sequence such that

$$E^1_{a,b} = H_b(\tilde{\mathscr{J}}^{t,r}_{*,a}) \to H_{a+b}(\tilde{\mathscr{J}}^{t,r}).$$

It is easy to see that

$$\tilde{\mathscr{J}}^{t,r}_{*,a} \simeq \mathscr{J}^{t,r-a} \otimes \Lambda^a(\mathrm{id}_{F \otimes G})$$

as a graded R-complex, so we have

$$E^1_{a,b} \simeq \begin{cases} 0 & (a > 0) \\ H_b(\mathscr{J}^{t,r}) & (a = 0) \end{cases},$$

since $\Lambda^a(\mathrm{id}_{F \otimes G})$ is homotopically trivial if $a > 0$ (see Lemma IV.1.7). Hence we have the second isomorphism.

We study the complex $\tilde{\mathscr{J}}^{t,r}$ for $r \geqslant t$ to calculate the Betti numbers of S/I_t. We denote the morphism id_F by $\mathrm{id}_F: F_1 \to F_0$ to distinguish the source $(F = F_1)$ and the target $(F = F_0)$. We choose ordered bases of F_0 and F_1

$X_0 = \{x_1 < \cdots < x_m\}$ and $X_1 = \{x'_1 < \cdots < x'_n\}$ so that $\mathrm{id}_F(x'_j) = x_j$. We let $X = X_0 \cup X_1$ and $X_0 < X_1$ (i.e., $\forall i, j\ x_i < x'_j$). For $\lambda \in \Omega^+$, we define

$$X_\lambda = \{S \in \mathrm{Tab}_\lambda X \mid S \text{ is row standard mod } X_1\}$$
$$(= \mathrm{Row}_\lambda\,(X, X_1), \text{ see chapter I, §3}).$$

For $S \in \mathrm{Tab}_\lambda X$ and $i \in \mathbb{N}$, we set $n_i(S) = \#\{j \mid S(i,j) \in X_1\}$ $(= n_i(S, X_1))$ and $n(S) = \sum_{i \in \mathbb{N}} n_i(S)$.

Similarly, we denote id_G by $\mathrm{id}_G : G_1 \to G_0$ and take bases $Y_0 = \{y_1 < \cdots < y_n\}$ and $Y_1 = \{y'_1 < \cdots < y'_n\}$ so that $\mathrm{id}_G(y'_j) = y_j$, $Y = Y_0 \cup Y_1$, and so that $Y_0 < Y_1$. We define $Y_\lambda = \mathrm{Row}_\lambda(Y, Y_1)$ for $\lambda \in \Omega^+$. For $T \in \mathrm{Tab}_\lambda\, Y$ and $i \in \mathbb{N}$, we set $n_i(T) = n_i(T, Y_1)$ and $n(T) = \sum_{i \in \mathbb{N}} n_i(T)$.

Let $\lambda \in \Omega^+$ and Z be a set. For $z_{i,j} \in Z$ $((i,j) \in \Delta_\lambda)$, we have a tableau $T \in \mathrm{Tab}_\lambda\, Z$ such that $T(i,j) = z_{i,j}$. We sometimes express T like a matrix,

$$T = \begin{matrix} z_{11} \cdots z_{1\lambda_1} \\ z_{21} \cdots z_{1\lambda_2} \\ \cdots \\ z_{q1} \cdots z_{q\lambda_q} \end{matrix},$$

where $q = \lg(\lambda)$. With $S \in \mathrm{Tab}_\lambda\, X$, we associate an element $X_S \in \Lambda_\lambda \mathrm{id}_F$. Sometimes S itself (or its matrix expression) will stand for X_S, if there is no danger of confusion. A similar convention will be applied to an element of $\mathrm{Tab}_\lambda\, Y$.

DEFINITION IV.1.5. For $l \in \mathbb{N}$, $i, j \in \mathbb{N}_0$, and $\lambda \in \Omega_r^+$, we define $L_{i,j}^{t,\lambda,l}$ to be the R-submodule of $\Lambda_\lambda \mathrm{id}_F \otimes \Lambda_\lambda \mathrm{id}_G$ generated by $\{S \otimes T \mid (S, T) \in X_\lambda \times Y_\lambda,\ n(S) = i,\ n(T) = j,\ n_l(S) + n_l(T) \leqslant \lambda_l - t\}$. We also define $L_{i,j}^{t,\lambda} = \sum_{l \in \mathbb{N}} L_{i,j}^{t,\lambda,l}$.

It is easy to see that $L^{t,\lambda} = \sum_{i,j} L_{i,j}^{t,\lambda}$ is a sub- (double) complex of $\Lambda_\lambda \mathrm{id}_F \otimes \Lambda_\lambda \mathrm{id}_G$.

DEFINITION IV.1.6. For $\lambda \in \Omega_r^-$, we define

$$M_{i,j}^{t,\lambda} = \sum_{\mu \in \Omega_r^-, \mu \in \lambda} \theta_\mu(L_{i,j}^{t,\mu}) (\subset M^\lambda(\theta)(\mathrm{id}_F, \mathrm{id}_G))$$

and

$$\dot{M}_{i,j}^{t,\lambda} = \sum_{\mu \in \Omega_r^-, \mu > \lambda} \theta_\mu(L_{i,j}^{t,\mu}) (\subset \dot{M}^\lambda(\theta)(\mathrm{id}_F, \mathrm{id}_G)).$$

By the definition of θ (Chapter III), if $k, a, b \in \mathbb{N}_0$ and if $a + b \leqslant k - t$, then the image of $\Lambda^{k-a}F_0 \otimes D_a F_1 \otimes \Lambda^{k-b} G_0 \otimes D_b G_1$ by the morphism $\theta_k(\mathrm{id}_F, \mathrm{id}_G)$ is contained in $\tilde{\mathscr{J}}^{t,k}$. So it is easy to see that

LEMMA IV.1.7. *For $\lambda \in \Omega_r^-$, $M^{t,\lambda} = \sum_{i,j} M_{i,j}^{t,\lambda}$ is a sub- (double) complex of $\tilde{\mathscr{J}}^{t,r}$. And we have $M^{\omega_{t,r}} = \tilde{\mathscr{J}}^{t,r}$. Hence $\{M^{t,\lambda}\}_{\lambda \geqslant \omega_{t,r}}$ gives a filtration of $\tilde{\mathscr{J}}^{t,r}$.*

We define $s_F^k: \Lambda^k \mathrm{id}_F \to \Lambda^k \mathrm{id}_F$ for $k \in \mathbb{N}_0$ as follows. Let $S = x_{i_1} \cdots x_{i_{k-l}} x'_{j_1} \cdots x'_{j_l}$ be a single-rowed tableau in $X_{k \cdot \varepsilon_1}$. We define

$$s_F^k(S) = \begin{cases} x_{i_1} \cdots x_{i_{k-l-1}} x'_{j_1} \cdots x'_{j_l} x'_{i_{k-l}} & (\text{if } i_{k-l} \geqslant j_l) \\ 0 & (\text{if } i_{k-l} < j_l) \end{cases}.$$

Note that this definition of s_F^k depends on the choice of the ordered basis X.

LEMMA IV.1.8. *s_F^k is a chain deformation (i.e., $\partial \circ s_F^k + s_F^k \circ \partial = \mathrm{id}$ and ∂ is of degree $(0, 1)$). So $\Lambda^k \mathrm{id}_F$ is homotopically trivial.*

Proof. Straightforward computation.

Let $\lambda \in \Omega^+$ with $|\lambda| \geqslant 1$ and $\lg(\lambda) = q$. For $i \in \operatorname{Supp} \lambda$, we define $s_F^{\lambda, i}: \Lambda_\lambda \mathrm{id}_F \to \Lambda_\lambda \mathrm{id}_F$ to be the morphism which equals $(-1)^l \mathrm{id} \otimes s_F^{\lambda_i} \otimes \mathrm{id}$ on $[\Lambda_{(\lambda_1, \ldots, \lambda_{i-1})} \mathrm{id}_F]_l \otimes \Lambda^{\lambda_i} \mathrm{id}_F \otimes \Lambda_{(\lambda_{i+1}, \ldots, \lambda_q)} \mathrm{id}_F \subset \Lambda_\lambda \mathrm{id}_F$. It is easy to see that $s_F^{\lambda, i}$ is a chain deformation of $\Lambda_\lambda \mathrm{id}_F$. We denote $s_F^{\lambda, q}$ by s_F^λ. We define s_G^k, $s_G^{\lambda, i}$, and s_G^λ in a similar way. If $S \in X_\lambda$ and $i \in \operatorname{Supp} \lambda$, then $s_F^{k,i}(S) = 0$ or $s_F^{k,i}(S) = \pm S'$, where S' is a tableau in X_λ which satisfies $n_i(S') = n_i(S) + 1$ and $n_j(S') = n_j(S)$ for $j \neq i$. Hence, we have

LEMMA IV.1.9. *Let $a, b \in \mathbb{N}_0$ and $\lambda \in \Omega^+$. If $\lg(\lambda) \geqslant 2$ or $|\lambda| \geqslant t + a + b + 1$, then $s_F^\lambda \otimes \mathrm{id}_{\Lambda_\lambda \mathrm{id}_G}$ (resp. $\mathrm{id}_{\Lambda_\lambda \mathrm{id}_F} \otimes s_G^\lambda$) maps $L_{a,b}^{t,\lambda,1}$ into $L_{a+1,b}^{t,\lambda,1}$ (resp. $L_{a,b+1}^{t,\lambda,1}$). Hence $s_F^\lambda \otimes \mathrm{id}$ (resp. $\mathrm{id} \otimes s_G^\lambda$) is a chain deformation of the complex $L_{*,b}^{t,\lambda,1}$ (resp. $L_{a,*}^{t,\lambda,1}$) if $\lg(\lambda) \geqslant 2$.*

2. *Calculation of $\underline{H}_2(\tilde{\mathscr{J}}^t)$*

In this section, we show that $\underline{H}_2(\tilde{\mathscr{J}}^t)$ is R-free if $t \geqslant \min(m, n) - 2$.

LEMMA IV.2.1. *Let $\lambda \in \Omega^-$. If $\lambda_2 < t$ or $\lambda_1 = t$, then $M_{i,j}^{t,\lambda} = \dot{M}_{i,j}^{t,\lambda} + \theta_\lambda(L_{i,j}^{t,\lambda,1})$ for any $i, j \in \mathbb{N}_0$.*

Proof. If $\lambda_2 < t$, then $L_{i,j}^{t,\lambda} = L_{i,j}^{t,\lambda,1}$ from the definition of $L_{i,j}^{t,\lambda}$. Hence, this case is trivial. Let $\lambda_1 = t$. It suffices to show that $\theta_\lambda(S \otimes T) \in \dot{M}_{i,j}^{t,\lambda} + \theta_\lambda(L_{i,j}^{t,\lambda,1})$ for $S \in X_\lambda$ and $T \in Y_\lambda$ such that $n(S) = i$, $n(T) = j$, and that $^\exists l \geqslant 2$ $n_l(S) + n_l(T) \leqslant \lambda_l - t$. Since $\lambda_l \leqslant \lambda_1 = t$, $\lambda_l = t$ and $n_l(S) = n_l(T) = 0$. From (III.2.4), we have $\theta_\lambda(S \otimes T) = \pm \theta_\lambda(S' \otimes T')$ where

S' (resp. T') is the tableau obtained by swapping the first row and the lth row of S (resp. T). It is clear that $S' \otimes T' \in L^{t,\lambda,1}_{i,j}$. Q.E.D.

LEMMA IV.2.2. *Let $\lambda \in \Omega^-$ and $i, j \in \mathbb{N}_0$ with $i+j \leqslant 2$. We have $M^{t,\lambda}_{i,j} = \dot{M}^{t,\lambda}_{i,j} + \theta_\lambda(L^{t,\lambda,1}_{i,j})$ except for the case $i=j=1$, $\lambda_1 = t+1$, and $\lambda_2 = t$.*

Proof. It suffices to show that $\theta_\lambda(S \otimes T) \in \dot{M}^{t,\lambda}_{i,j} + \theta_\lambda(L^{t,\lambda,1}_{i,j})$ for $S \in X_\lambda$ and $T \in Y_\lambda$ and $T \in Y_\lambda$ such that $n(S) = i$, $n(T) = j$, $n_1(S) + n_1(T) > \lambda_1 - t$, and $^\exists l \geqslant 2\ n_l(S) + n_l(T) \leqslant \lambda_l - t$. From (IV.2.1), we may assume that $\lambda_1 > t$ and that $\lambda_2 \geqslant t$. Hence we have $1 \leqslant \lambda_1 - t < n_1(S) + n_1(T) \leqslant i+j \leqslant 2$. So $\lambda_1 = t+1$, $i+j=2$, $n_1(S) = i$, and $n_1(T) = j$. Note that $n_k(S) = n_k(T) = 0$ for $k \geqslant 2$. Hence we have $S \otimes T \in L^{t,\lambda,2}_{i,j}$. If $\lambda_2 = \lambda_1 = t+1$, we may proceed as in (IV.2.1). So we may assume that $\lambda_2 = t$. By assumption, we have only to treat the cases $(i,j) = (2,0)$ and $(i,j) = (0,2)$. Let us consider the case $(i,j) = (2,0)$. We put

$$S' = \begin{array}{l} S(1,1) \cdots S(1,t-1) \\ S(1,t)\, S(1,t+1)\, S(2,1) \cdots S(2,t-1)\, S(2,t) \in \mathrm{Tab}_{\lambda - 2\cdot\alpha_1} X \\ S(3,1) \cdots S(3,\lambda_3) \\ \quad\cdots \end{array}$$

so that $\tilde{\square}^{\lambda-2\cdot\alpha_1}_{\lambda}(S') = S + \sum_U \pm U$, where $U \in X_\lambda$ with $n_1(U) \leqslant 1$. From (III.2.6), $\theta_\lambda(\tilde{\square}^{\lambda-2\cdot\alpha_1}_{\lambda}(S') \otimes T) = \theta_{\lambda-2\cdot\alpha_1}(S' \otimes \square^{\lambda}_{\lambda-2\cdot\alpha_1}(T))$. From (III.2.4), $\theta_{\lambda-2\cdot\alpha_1}(S' \otimes \square^{\lambda}_{\lambda-2\cdot\alpha_1}(T)) \in M^{\widetilde{\widetilde{(\lambda-2\cdot\alpha_1)}}}_{2,0}(\theta) = M^{t,\widetilde{\widetilde{(\lambda-2\cdot\alpha_1)}}}_{2,0}$. On the other hand, $n_1(U) \leqslant 1$ implies that $U \otimes T \in L^{t,\lambda,1}_{2,0}$ for each U. Hence, $S \in \dot{M}^{t,\lambda}_{2,0} + \theta_\lambda(L^{t,\lambda,1}_{2,0})$.

The case $(i,j) = (0,2)$ is quite similar, so we omit it. Q.E.D.

Now we fix $r \in \mathbb{N}$ with $r \geqslant t+2$ and $\lambda \in \Omega^-_r$ with $\lambda_1 \geqslant t$. We want to prove

PROPOSITION IV.2.3. *Let r and λ be as above. Then we have $H_2(M^{t,\lambda}/\dot{M}^{t,\lambda}) = 0$, except for the following three cases:*

(i) $\lambda = (t+2)$.

(ii) $\lambda = (t+1, t)$, $\min(m,n) \geqslant |\lambda| = 2t+1$.

(iii) $r \leqslant 2t$, $\lambda = (t, r-t)$, $\min(m,n) \geqslant |\lambda| = r$.

We may assume that $m \leqslant n$ since $S_{|\lambda|}(T_{\mathrm{id}_F,\mathrm{id}_G})$: $S_{|\lambda|}(\mathrm{id}_F \otimes \mathrm{id}_G) \to S_{|\lambda|}(\mathrm{id}_G \otimes \mathrm{id}_F)$ induces isomorphism $M^{t,\lambda}(F,G) \xrightarrow{\simeq} M^{t,\lambda}(G,F)$. Since $M^{t,\lambda}/\dot{M}^{t,\lambda}$ is a double complex, we can associate the usual spectral sequence whose E^1-term is $E^1_{a,b} = H_b(M^{t,\lambda}_{*,a}/\dot{M}^{t,\lambda}_{*,a}) \to H_{a+b}(M^{t,\lambda}/\dot{M}^{t,\lambda})$. To prove the proposition, it is sufficient to show that

(a) $E^2_{0,2}=0$

(b) $E^2_{1,1}=0$

(c) $E^1_{2,0}=0.$

LEMMA IV.2.4. *$E^1_{2,0}=0$ for any $\lambda\in\Omega_r^-$ with $\lambda_1\geqslant t$ and $\lambda\neq(t+2)$.*

Proof. From (IV.1.9), $s^\lambda_F\otimes\mathrm{id}$ maps $L^{t,\lambda,1}_{0,2}$ into $L^{t,\lambda,1}_{1,2}$. From (IV.1.8), $\partial^\lambda_F\circ s^\lambda_F=\mathrm{id}$, where $\partial^\lambda_F=\partial^{A_\lambda \mathrm{id}_F}$. Hence, $\partial^\lambda_F\colon L^{t,\lambda,1}_{1,2}\to L^{t,\lambda,1}_{0,2}$ is surjective. Since $M^{t,\lambda}_{0,2}=\dot{M}^{t,\lambda}_{0,2}+\theta_\lambda(L^{t,\lambda,1}_{0,2})$ from (IV.2.2), $\bar{\partial}_F\colon M^{t,\lambda}_{1,2}/\dot{M}^{t,\lambda}_{1,2}\to M^{t,\lambda}_{0,2}/\dot{M}^{t,\lambda}_{0,2}$ is surjective, where $\bar{\partial}_F$ is the boundary map of $M^{t,\lambda}_{*,2}/\dot{M}^{t,\lambda}_{*,2}$. Hence, $E^1_{2,0}=0$ as we desired.

LEMMA IV.2.5. *If $\lambda, m,$ and n are as in* (IV.2.3) *and $m\leqslant n$, then $M^{t,\lambda}_{1,1}=\dot{M}^{t,\lambda}_{1,1}+\partial_F(M^{t,\lambda}_{2,1})+\theta_\lambda(L^{t,\lambda,1}_{1,1})$, where ∂_F is the boundary map of $\tilde{\mathscr{J}}^{t,r}_{*,1}$.*

Proof. From (IV.2.2), we may assume that $\lambda_1=t+1$ and $\lambda_2=t$. It is sufficient to prove that $\theta_\lambda(S\otimes T)\in\dot{M}^{t,\lambda}_{1,1}+\partial_F(M^{t,\lambda}_{2,1})+\theta_\lambda(L^{t,\lambda,1}_{1,1})$ for $S\in X_\lambda$ with $n(S)=1$ and $T\in Y_\lambda$ with $n(T)=1$.

If $\lg(\lambda)\geqslant 3$, $S'=s^\lambda_F\circ\partial^\lambda_F(S)$ is a tableau in X_λ with $n_1(S')=0$. In this case, $\theta_\lambda(S\otimes T)=\theta_\lambda(s^\lambda_F\circ\partial^\lambda_F(S)\otimes T)+\theta_\lambda(\partial^\lambda_F\circ s^\lambda_F(S)\otimes T)=\theta_\lambda(S'\otimes T)+\partial_F\circ\theta_\lambda(s^\lambda_F(S)\otimes T)\in\theta_\lambda(L^{t,\lambda,1}_{1,1})+\partial_F(M^{t,\lambda}_{2,1})$, since $S'\otimes T\in L^{t,\lambda,1}_{1,1}$ and $s^\lambda_F(S)\otimes T\in L^{t,\lambda,1}_{2,1}+L^{t,\lambda,2}_{2,1}$.

So we may assume that $\lambda=(t+1,t)$ and $m<|\lambda|$, from the assumption of (IV.2.3) and our assumption $m\leqslant n$. If $n_1(S)+n_1(T)\leqslant 1$, there is nothing to prove, because $S\otimes T\in L^{t,\lambda,1}_{1,1}$. So we assume that $n_1(S)=n_1(T)=1$. Since $m<|\lambda|$, $n_{\{1,2\}}(S,\{x_k,x'_k\})\geqslant 2$ for some $k\in\{1,\ldots,m\}$. We put

$$S=\begin{matrix}x_{i_1}\cdots x_{i_t}x'_c\\ x_{j_1}\cdots x_{j_t}\end{matrix}.$$

If $c\neq k$, then $k=i_u$ for some $u\in\{1,\ldots,t\}$. In this case, if we put

$$S'=\begin{matrix}x_{i_1}\cdots\hat{x}_{i_u}\cdots x_{i_t}x_cx'_k\\ x_{j_1}\cdots\cdots\cdots x_{j_t}\end{matrix},\qquad U=\begin{matrix}x_{i_1}\cdots\hat{x}_{i_u}\cdots x_{i_t}x'_cx'_k\\ x_{j_1}\cdots\cdots\cdots x_{j_t}\end{matrix},$$

then $S=(-1)^{t-u}(\partial^\lambda_F(U)-S')$, where the symbols $\hat{}$ mean that we omit the entries x_{i_u} below these symbols (we will use this convention later). Hence, $\theta_\lambda(S\otimes T)\pm\theta_\lambda(S'\otimes T)\in\partial_F M^{t,\lambda}_{2,1}$. Therefore, we may replace S by the row standarization of S', so we may assume that $c=k$. If $k=i_u$ for some u, then $S=(-1)^{t-u}\,\partial^\lambda_F(U)$. In this case, $\theta_\lambda(S\otimes T)\in\partial_F M^{t,\lambda}_{2,1}$. If $k\neq i_u$ for any u, there is some v such that $k=j_v$. We put

$$V=\begin{matrix}x_{i_1}\cdots\cdots\cdots\cdots x_{i_t}x_kx'_k\\ x_{j_1}\cdots\hat{x}_{j_v}\cdots x_{j_t}\end{matrix}\in\mathrm{Tab}_{(t+2,t-1)}X.$$

It is easy to see that $S = \pm \Box_\lambda(V) + \sum \pm S'' + W$, where S'' is a tableau in X_λ such that $S''(1, t+1) = x'_k$ and $S''(1, u) = x_k$ for some u, and $W \in X_\lambda$ such that $n_1(W) = 0$. Hence, $\theta_\lambda(S \otimes T) \in \dot{M}^{t,\lambda}_{1,1} + \partial_F(M^{t,\lambda}_{2,1}) + \theta_\lambda(L^{t,\lambda,1}_{1,1})$ and we have completed the proof. Q.E.D.

LEMMA IV.2.6. *If $a, b \in \mathbb{N}_0$ with $a+b \leqslant 3$, then $\dot{M}^{t,\lambda}_{a,b} + \theta_\lambda(L^{t,\lambda,1}_{a,b})$ is generated by $\dot{M}^{t,\lambda}_{a,b} \cup \{\theta_\lambda(S \otimes T) \mid S \in X_\lambda,\ T \in Y_\lambda,\ S \otimes T \in L^{t,\lambda,1}_{a,b}$, S is standard* mod X_1, *and T is standard* mod $Y_1\}$.

Proof. Let $S \in X_\lambda$, $T \in Y_\lambda$, and $S \otimes T \in L^{t,\lambda,1}_{a,b}$. Assume that S is not column-standard mod X_1, so that $S(p, q) \geqslant S(p+1, q)$ for some p, q. An argument similar to the proof of [2, Lemma II.2.15] guarantees that there are some tableaux $S_i \in X_\lambda$ and $U_{\mu,j} \in X_\mu$ $(\mu = \lambda + k \cdot \alpha_p,\ 1 \leqslant k \leqslant \lambda_{p+1})$ such that $S_i < S$ (for the definition of the relation $<$ between tableaux, see (I.3.7)), $n(S_i) = n(U_{\mu,j}) = n(S)$ and that $S - \sum_i c_i \cdot S_i = \sum_{\mu,j} c_{\mu,j} \cdot \Box_\lambda(U_{\mu,j})$, where c_i, $c_{\mu,j} \in \mathbb{Z}$. If $S(p+1, q) \in X_0$ or $p \geqslant 2$, then we can take $U_{\mu,j}$ to be satisfying $n_1(U_{\mu,j}) \leqslant n_1(S)$ from (I.3.9). So in this case, it is easy to check that $U_{\mu,j} \otimes \tilde{\Box}^\lambda_\mu(T) \in L^{t,\mu,1}_{a,b}$. If $p = 1$ and $S(2, q) \in X_1$, then $S(1, q) \in X_1$. Hence, $q \geqslant t+1$. In this case, $U_{\mu,j} \otimes \tilde{\Box}^\lambda_\mu(T) \in L^{t,\mu,1}_{a,b} + L^{t,\mu,2}_{a,b}$, since $\mu_1 + \mu_2 \geqslant 2t+2$ and $a+b \leqslant 3$. In case, we have $\theta_\lambda(S \otimes T) - \sum_i c_i \cdot \theta_\lambda(S_i \otimes T) \in \dot{M}^{t,\lambda}_{a,b}$ from (III.2.6). If S_i is not standard, we may repeat this procedure. However, since X_λ is a finite set, we must ultimately arrive at standard tableaux. So we may take S to be standard. Similar argument guarantees that we may also take T to be standard. This completes the proof of this lemma. Q.E.D.

LEMMA IV.2.7. *If $\lambda \in \Omega^-_r$ with $r \geqslant t+2$ and $\lambda \neq (t+2)$, then $E^2_{0,2} = 0$ (i.e., $H^G_0(H^F_2(M^{t,\lambda}_{*,*}/\dot{M}^{t,\lambda}_{*,*})) = 0$).*

Proof. From (IV.2.2) and (IV.2.6), any element of $H_2(M^{t,\lambda}_{*,0}/\dot{M}^{t,\lambda}_{*,0})$ is represented by $A = \sum_{S,T} c_{S,T} \cdot \theta_\lambda(S \otimes T)$, where $S \in X_\lambda$, $T \in Y_\lambda$, $S \otimes T \in L^{t,\lambda,1}_{2,0}$, S is standard mod X_1, T is standard mod Y_1, and $c_{S,T} \in R$. Since $\partial_F(A) = \sum_T \theta_\lambda((\sum_S c_{S,T} \partial^\lambda_F S) \otimes T) \in \dot{M}^{t,\lambda}_{1,0}$, we have $\sum_S c_{S,T} \partial^\lambda_F S \in \operatorname{Im} \Box_\lambda$ for each T, from (III.2.7) and (I.3.11). Hence, we can write $\sum_S c_{S,T} \partial^\lambda_F S = \sum_{\mu \in S_\Box(\lambda)} \Box^\mu_\lambda(a^T_\mu)$, where $a^T_\mu = \sum_U c_{U,\mu} \cdot U$ with $U \in X_\mu$ and $n(U) = 1$. It is easy to check that we can take each U so that $n_1(U) = 0$, if $n_1(S) = 0$ for each S, from (I.3.9). In particular, we have $n_1(T) + n_1(U) \leqslant \lambda_1 - t$ for each T and U which appears in a^T_μ. Consider the element $\tilde{A} = \sum_{S,T} c_{S,T} \cdot \theta_\lambda(S \otimes s^\lambda_G T)$. From our assumption on λ, $\tilde{A} \in M^{t,\lambda}_{2,1}$, since $S \otimes s^\lambda_G T \in L^{t,\lambda,1}_{2,1}$ for each S and T from (IV.1.9). From (III.2.6), $\partial_F \tilde{A} = \sum_T \sum_\mu \sum_U c_{U,\mu} \cdot \theta_\lambda(\Box^\mu_\lambda(U) \otimes s^\lambda_G(T)) = \sum_{T,\mu,U} c_{U,\mu} \cdot \theta_\mu(U \otimes \tilde{\Box}^\lambda_\mu s^\lambda_G(T))$. If we write $\tilde{\Box}^\lambda_\mu s^\lambda_G(T) = \sum_i \pm T_{\mu,i}$ so that $T_{\mu,i} \in Y_\mu$ with $n(T_{\mu,i}) = 1$, it is clear that $n_1(T_{\mu,i}) \leqslant n_1(T) + \mu_1 - \lambda_1$. Hence, $n_1(U) + n_1(T_{\mu,i}) \leqslant n_1(U) + n_1(T) + \mu_1 - \lambda_1 \leqslant \mu_1 - t$. So we have $\partial_F \tilde{A} \in \dot{M}^{t,\lambda}_{1,1}$. In other words, $\tilde{A}$ represents an element of $H_2(M^{t,\lambda}_{*,1}/\dot{M}^{t,\lambda}_{*,1})$. On the other hand, it is easy to see that $\partial_G \tilde{A} = A$, from (IV.1.8). So A represents 0 in $E^2_{0,2}$. This completes the proof of this lemma. Q.E.D.

We are now ready to complete the proof of (IV.2.3).

LEMMA IV.2.8. *If λ, m, and n are as in* (IV.2.3) *with $m \leqslant n$, then* $E^2_{1,1} = H^G_1(H^F_1(M^{t,\lambda}_{*,*}/\dot{M}^{t,\lambda}_{*,*})) = 0$.

Proof. From (IV.2.5) and (IV.2.6), any element of $E^2_{1,1}$ is represented by $A = \sum_{S,T} c_{S,T} \cdot \theta_\lambda(S \otimes T)$, where $S \in X_\lambda$, $T \in Y_\lambda$, $S \otimes T \in L^{t,\lambda,1}_{1,1}$, S is standard mod X_1, T is standard mod Y_1, and $c_{S,T} \in R$. So we have $\sum_S c_{S,T} \cdot \partial^\lambda_F S \in \operatorname{Im} \square_\lambda$ for each T as in the proof of (IV.2.7). Assume that $\lg(\lambda) \geqslant 2$. If $n_1(T) = 1$, we replace T by $s^\lambda_G \partial^\lambda_G(T)$. We may do so, since $T - s^\lambda_G \partial^\lambda_G(T) = \partial^\lambda_G s^\lambda_G(T)$ and $\sum_S c_{S,T} \cdot \theta_\lambda(S \otimes \partial^\lambda_G s^\lambda_G(T))$ represents 0 in $E^2_{1,1}$. There exists an element $T' \in Y_\lambda$ such that $s^\lambda_G \partial^\lambda_G(T) = \pm T'$. From the definition of s^λ_G and our assumption on λ, we have $n_1(T') = 0$. T' is by no means standard, but we can replace it by a linear combination of standard (mod Y_1) tableau T'' with $n_1(T'') = 0$ as in the proof of (IV.2.6). Therefore, we may assume that $n_1(T) = 0$ for all T, or $\lg(\lambda) = 1$.

We can write $\sum_S c_{S,T} \partial^\lambda_F S = \sum_{\mu \in S_\square(\lambda)} \square^\mu_\lambda(a^T_\mu)$, where $a^T_\mu \in \Lambda_\mu F$. If $(r) \in S_\square(\lambda)$ and $a^T_{(r)} \neq 0$, then $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geqslant t+1$ and $\lambda_2 > 0$ from our exception (iii). Now consider the elements $b^T_1 = \sum_S c_{S,T} \cdot S \otimes T \in L^{t,\lambda,1}_{1,1}$ and $b^T_2 = \sum_{\mu \in S_\square(\lambda)} \square^\mu_\lambda(s^F_\mu(a^T_\mu)) \otimes T$. It is easy to see that $b^T_2 \in L^{t,\lambda,1}_{1,1}$, from the above remark. Note that $\theta_\lambda(b^T_2) \in \dot{M}^{t,\lambda}_{1,1}$ from (III.2.6) and our assumption $r \geqslant t+2$. Since $\partial^\lambda_F \otimes \mathrm{id}(b^T_1) = \partial^\lambda_F \otimes \mathrm{id}(b^T_2)$, we have $b^T_1 = b^T_2 + ((\partial^\lambda_F \circ s^\lambda_F) \otimes \mathrm{id})(b^T_1 - b^T_2)$. But we have $s^\lambda_F \otimes \mathrm{id}(b^T_1 - b^T_2) \in L^{t,\lambda,1}_{2,1}$ from (IV.1.9). Hence, $A = \sum_T \theta_\lambda(b^T_1)$ is contained in $\dot{M}^{t,\lambda}_{1,1} + \partial_F(M^{t,\lambda}_{2,1})$. This means that A represents 0 in $E^2_{1,1}$ and we have completed the proof. Q.E.D.

In Section 6 of Chapter II, we introduced the definition of the linear complex $\mathbb{X}^t$. By definition, the degree $t+r$ component $(X^t_{r+1})_{t+r}$ of X^t_{r+1} can be identified with $H_r(\mathscr{J}^{t,t+r})$ (remember that $\phi^S_t \colon \Lambda^t F \otimes \Lambda^t G \to I_{t,t}$ is an isomorphism).

LEMMA IV.2.9. *Let F and G be finite free R-modules. Then the complex $X^t_3 \to X^t_2 \to X^t_1 \to S \to 0$ is a universally free S-complex* (*i.e., the underlying R-complex is universally free*).

Proof. From Section 5 of Chapter II, it is easy to see that $[\underline{\mathrm{Tor}}^S_{i+1}(S/I_t, S/I_1)]_r \simeq H_i(\mathscr{J}^{t,r})$ is zero, if $i \leqslant 1$ and $r \neq t+i$. From (I.3.12) and (IV.1.2), if $r \leqslant 2$, $(X^t_{r+1})_{t+r} \simeq H_r(\mathscr{J}^{t,t+r})$ is universally free, and the inclusion map $(X^t_{r+1})_{t+r} \subsetneq \Lambda^r \otimes I_{t,t}$ is universal, since rank $H_r(\mathscr{J}^{t,t+r}) = \sum_i (-1)^{r+i-1}$ rank $\mathscr{J}^{t,t+r}_i$ is independent of the characteristic. It is easy to check the universality of the boundary maps from the definition of the boundary maps of $\mathbb{X}^t$. Q.E.D.

THEOREM IV.2.10. *Let F and G be finite free R-modules with* rank $F = m$ *and* rank $G = n$. *If $t, r \in \mathbb{N}$, $t+2 \geqslant \min(m, n)$, and $t+2 \neq r$, then $H_2(\mathcal{J}^{t,r}(F, G)) = 0$. Moreover, we have $H_2(\mathcal{J}^{t,t+2}) \simeq (X_3^t)_{t+2}$ is universally free. Hence, the third Betti number of S/I_t is independent of the characteristic.*

Proof. Since $\mathcal{J}_2^{t,r} = 0$ for $r < t+2$, it is clear that $H_2(\mathcal{J}^{t,r}) = 0$ for $r < t+2$. Consider the case $r > t+2$. From (IV.1.7), $\{M^{t,\lambda}\}_{\lambda \in \Omega_r^-, \lambda_1 \geqslant t}$ gives a filtration of $\tilde{\mathcal{J}}^{t,r}$. From (IV.2.3), we have $H_2(M^{t,\lambda}/\dot{M}^{t,\lambda}) = 0$ for $\lambda \in \Omega_r^-$ with $\lambda_1 \geqslant t$, since $r > t+2 \geqslant \min(m, n)$. Hence, $H_2(\mathcal{J}^{t,r}) = H_2(\tilde{\mathcal{J}}^{t,r}) = 0$ if $r > t+2$. So the theorem follows from (IV.2.9). Q.E.D.

COROLLARY IV.2.11. *If $t \leqslant \min(m, n) - 2$, then we have the exact sequence*

$$X_3^t(m, n) \to X_2^t(m, n) \to X_1^t(m, n) \to I_t \to 0, \qquad (*)$$

where $X^t = X^t(m, n)$ is the complex defined in (II.6.3).

Proof. Since $(*)$ is universally free complex, we may assume that R is a prime field $\mathbb{F}_p$.

Let $P. \to I_t \to 0$ be a minimal free resolution. From (II.5.2) and (IV.2.9),

$$P_3 \to P_2 \to P_1 \to I_t \to 0 \qquad (**)$$

is linear. By (II.6.5), $(*)$ is isomorphic to $(**)$. Hence, $(*)$ is exact.

COROLLARY IV.2.12. *If $t \leqslant \min(m, n) - 2$, then $X_4^t(m, n)$ is a universally free $GL(F) \times GL(G)$-module.*

Proof. Clear.

3. *Partial Calculation of $H_3(\mathcal{J}^{t,r})$*

Let $t, r \in \mathbb{N}$ with $1 \leqslant t \leqslant \min(m, n)$ and $r \geqslant t+3$.

PROPOSITION IV.3.1. *Let $\lambda \in \Omega_r^-$. If one of the following conditions is true, then $H_3(M^{t,\lambda}/\dot{M}^{t,\lambda}) = 0$.*

(i) $\lg(\lambda) = 1$, *and* $r \geqslant t+4$.

(ii) $\lambda_1 = t$ *or* $\lambda_2 < t$, *and one of these conditions is true*:

(a) $\lg(\lambda) \geqslant 3$,

(b) $\lg(\lambda) = 2$, $m = \text{rank } F \leqslant t+2$, *and* $n = \text{rank } G \leqslant t+2$.

Proof. We use the spectral sequence argument as in the proof of (IV.2.3). First, note that any element of $M^{t,\lambda}/\dot{M}^{t,\lambda}$ is represented by an element of $\theta_\lambda(L^{t,\lambda,1})$, from (IV.2.1). So $E^1_{3,0}=0$ and $E^1_{2,1}=0$ are proved quite similarly to the proofs of (IV.2.4) and (IV.2.8), respectively (note that if $(r)\in S_\square(\lambda)$, then $\lg(\lambda)=2$ so that $\Lambda^r F=0$). Hence, any element of $H_3(M^{t,\lambda}/\dot{M}^{t,\lambda})$ is represented by an element of $\theta_\lambda(L^{t,\lambda,1}_{2,1}+L^{t,\lambda,1}_{3,0})$.

Let $A\in\theta_\lambda(L^{t,\lambda,1}_{2,1})$ and $B\in\theta_\lambda(L^{t,\lambda,1}_{3,0})$, with $\partial(A+B)\in\dot{M}^{t,\lambda}$. It is sufficient to show that $A+B\in\dot{M}^{t,\lambda}+\partial(M^{t,\lambda})$. There is an element $b\in L^{t,\lambda,1}_{3,0}$ such that $\theta_\lambda(b)=B$. Since $(\mathrm{id}\otimes s^\lambda_G)(b)\in L^{t,\lambda,1}_{3,1}$, from (IV.1.9), $\partial\circ\theta_\lambda((\mathrm{id}\otimes s^\lambda_G)(b))=\partial_F\circ\theta_\lambda((\mathrm{id}\otimes s^\lambda_G)(b))-B$ is contained in $\partial(M^{t,\lambda})$. Since $\partial_F\circ\theta_\lambda((\mathrm{id}\otimes s^\lambda_G)(b))\in\theta_\lambda(L^{t,\lambda,1}_{2,1})$, we may assume that $B=0$. We may assume that $A=\theta_\lambda(a)$, with $a=\sum_{S,T}c_{S,T}\cdot S\otimes T$, where $S\in X_\lambda$, $n(S)=2$, $T\in Y_\lambda$, $n(T)=1$, S is standard mod X_1, T is standard mod Y_1, and $S\otimes T\in L^{t,\lambda,1}_{2,1}$. Note that $\sum_S c_{S,T}\cdot\partial^\lambda_F(S)\in\operatorname{Im}\square_\lambda$ for each T. For each T, we may write $\partial^\lambda_G(T)=h(T)+\sum_{\mu\in S_\square(\lambda)}\square^\mu_\lambda(b_{\mu,T})$, where $h(T)$ is a linear combination of standard tableaux, and $b_{\mu,T}\in\Lambda_\mu G$. We may assume that each $\mu\in S_\square(\lambda)$ is not of length one. Consider the element $a'=\sum_{S,T,\mu}c_{S,T}\cdot S\otimes\square^\mu_\lambda(s^\lambda_G(b_{\mu,T}))$. It is easy to see that $\theta_\lambda(a')\in\dot{M}^{t,\lambda}_{2,1}$. We have $(\mathrm{id}\otimes\partial^\lambda_G)(a-a')=\sum_{S,T}c_{S,T}\cdot S\otimes h(T)$. But $\theta_\lambda\circ(\mathrm{id}\otimes\partial^\lambda_G)(a-a')=\partial_G\theta_\lambda(a-a')\in\dot{M}^{t,\lambda}_{2,0}\subset\dot{M}^\lambda(\theta)$. Hence $(\mathrm{id}\otimes\partial^\lambda_G)(a-a')=0$ from (III.2.7) and (I.3.11). So we have $a=a'+(\mathrm{id}\otimes(\partial^\lambda_G\circ s^\lambda_G))(a-a')$. Set $b_T=s^\lambda_G(T-\square^\mu_\lambda(s^\lambda_G(b_{\mu,T})))$ for each T. We have $(\mathrm{id}\otimes s^\lambda_G)(a-a')=\sum_{S,T}c_{S,T}\cdot S\otimes b_T$. So it is not so difficult to show that $\partial_F\circ\theta_\lambda\circ(\mathrm{id}\otimes s^\lambda_G)(a-a')\in\dot{M}^{t,\lambda}_{1,2}$. Hence, we have $A=\theta_\lambda(a')+\partial\circ\theta_\lambda\circ(\mathrm{id}\otimes s^\lambda_G)(a-a')-\partial_F\circ\theta_\lambda\circ(\mathrm{id}\otimes s^\lambda_G(a-a')\in\dot{M}^{t,\lambda}+\partial(M^{t,\lambda})$. This completes the proof of the proposition. Q.E.D.

From the above proof, we have $E^2_{1,2}=E^3_{0,3}=0$. More precisely, it holds that $E^2_{0,3}=0$ (left to the reader).

PROPOSITION IV.3.2. *If $m,n\leqslant t+3$ and $t+4\leqslant r\leqslant 2t$, then $H_3(\mathscr{J}^{t,r})=0$.*

Proof. If $\lambda\in\Omega^-_r$ with $\lambda_1\geqslant t$, then λ,m,n and t satisfies one of the conditions in (IV.3.1). So the proposition is clear. Q.E.D.

V. RESOLUTIONS OF DETERMINANTAL IDEALS: n-MINORS OF $(n+2)$-SQUARE MATRICES

Let n be an integer satisfying $n\geqslant 2$. In this chapter we show the existence of minimal free resolutions of the ideals defined by n-minors of the generic $(n+2)$-square matrices. (When $n=1$, one can construct the minimal free resolution from Koszul complex.) Throughout this chapter R is the integers $\mathbb{Z}$ or a field k, and S is the polynomial ring over R with variables x_{ij} for

$i, j = 1, 2, ..., n+2$. Let both F and G be finitely generated R-free modules of rank $n+2$.

1. *Universal Freeness of the Linear Complex* $\mathbb{X}_{\cdot}^{n}(n+2, n+2)$

We have already seen that $X_i^n(n+2, n+2)$ is universally free for $i = 1, 2, 3, 4$. In this section we shall show that $X_5^n(n+2, n+2)$ is also universally free. Since $\mathbb{X}_{\cdot}^{n+1}(n+2, n+2)$ is a linear complex (cf. Section 6 of Chapter II) of the Gulliksen–Negård complex [13], we have $X_4^{n+1} = 0$. Then X_5^n is isomorphic to Z_5^n by Remark II.6.5. Therefore we have only to show that Z_5^n is universally free. Let ∂ be the map

$$\Lambda^4(F\otimes G)\otimes \Lambda^n F\otimes \Lambda^n G \xrightarrow{\partial} \Lambda^3(F\otimes G)L_\lambda F\otimes L_\lambda G,$$

where $\lambda = (n, 1)$. (∂ is the composite map

$$\begin{aligned}\Lambda^4(F\otimes G)\otimes \Lambda^n F\otimes \Lambda^n G &\xrightarrow{\Delta\otimes 1} \Lambda^3(F\otimes G)\otimes F\otimes G\otimes \Lambda^n F\otimes \Lambda^n G\\ &\xrightarrow{T} \Lambda^3(F\otimes G)\otimes \Lambda^n F\otimes F\otimes \Lambda^n G\otimes G\\ &\xrightarrow{1\otimes d_\lambda\otimes d_\lambda} \Lambda^3(F\otimes G)\otimes L_\lambda F\otimes L_\lambda G,\end{aligned}$$

where T is an appropriate twisting map.) By the definition of Z_5^n, it is easy to see that Z_5^n is universally free if and only if $\operatorname{rank}_R(\operatorname{Ker}(\partial))$ does not depend on R. It suffices to show the following lemma.

LEMMA V.1.1. *There exists a homomorphism δ such that the following sequence is exact*:

$$\begin{aligned}0 \longrightarrow \Lambda^{n+2}F\otimes D_2G\otimes \Lambda^{n+2}G\otimes D_2G\\ \xrightarrow{\delta} \Lambda^4(F\otimes G)\otimes \Lambda^n F\otimes \Lambda^n G \longrightarrow \Lambda^3(F\otimes G)\otimes L_\lambda F\otimes L_\lambda G.\end{aligned}$$

Proof. Let $\{f_1, ..., f_{n+2}\}$ and $\{g_1, ..., g_{n+2}\}$ be free bases of F and G, respectively.

First we define the map δ. Consider the composite map

$$\begin{aligned}&\Lambda^{n+2}F\otimes D_2G\otimes \Lambda^{n+2}G\otimes D_2F\\ &\xrightarrow{\Delta\otimes 1\otimes\Delta\otimes 1} \Lambda^2F\otimes \Lambda^n F\otimes D_2G\otimes \Lambda^2G\otimes \Lambda^n G\otimes D_2F\\ &\xrightarrow{T} (\Lambda^2F\otimes D_2G)\otimes (D_2F\otimes \Lambda^2G)\otimes \Lambda^n F\otimes \Lambda^n G\\ &\xrightarrow{\psi^\Lambda\otimes\phi^\Lambda\otimes 1\otimes 1} \Lambda^2(F\otimes G)\otimes \Lambda^2(F\otimes G)\otimes \Lambda^n F\otimes \Lambda^n G\\ &\xrightarrow{m\otimes 1\otimes 1} \Lambda^4(F\otimes G)\otimes \Lambda^n F\otimes \Lambda^n G,\end{aligned}$$

where ψ^Λ and ϕ^Λ are the pairings defined in Section 1 of Chapter III. This composite map is denoted by δ. It is easy to check that ∂ and δ are $GL(F)\times GL(G)$-morphisms.

Next we show that $\partial \circ \delta = 0$. It is obvious that we have only to show $\partial \circ \delta = 0$ when R is the field of the rationals $\mathbb{Q}$. Therefore suppose $R = \mathbb{Q}$. Then $\Lambda^{n+2} F \otimes D_2 G \otimes \Lambda^{n+2} G \otimes D_2 F$ is an irreducible polynomial $GL(F) \times GL(G)$-representation. Hence $\partial \circ \delta$ is either injective or 0. So we have only to show that it is not injective. By direct computation we get

$$\partial \circ \delta(f_1 \wedge \cdots \wedge f_{n+2} \otimes g^{(2)}_{n+2} \otimes g_1 \wedge \cdots \wedge g_{n+2} \otimes f^{(2)}_{n+2}) = 0.$$

This implies $\partial \circ \delta = 0$.

We now prove exactness. Define the sets $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$ as

$$\mathscr{B}_1 = \left\{ \begin{array}{l|l} f_{i_1} \wedge \cdots \wedge f_{i_{n+2}} \otimes g_{j_{n+3}} \# g_{j_{n+4}} & i_1 < \cdots < i_{n+2}, \\ \quad \otimes g_{j_1} \wedge \cdots \wedge g_{j_{n+2}} \otimes f_{i_{n+3}} \# f_{i_{n+4}} & i_{n+3} \leqslant i_{n+4}, \\ & j_1 < \cdots < j_{n+2}, \\ & j_{n+3} \leqslant j_{n+4}, \end{array} \right\}$$

$$\mathscr{B}_2 = \left\{ \begin{array}{l|l} & i_5 < \cdots < i_{n+4}, \\ (f_{i_1} \otimes g_{j_1}) \wedge (f_{i_2} \otimes g_{j_2}) & j_5 < \cdots < j_{n+4}. \\ \wedge (f_{i_3} \otimes g_{j_3}) \wedge (f_{i_4} \otimes g_{j_4}) & i_k < i_{k+1} \\ & \quad \text{or} \\ \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} & i_k = i_{k+1} \text{ and } j_k < j_{k+1} \\ & \quad \text{for } k = 1, 2, 3 \end{array} \right\}$$

$$\mathscr{B}_3 = \left\{ \begin{array}{l|l} & i_4 < \cdots < i_{n+3}, \quad i_4 \leqslant i_{n+4}, \\ (f_{i_1} \otimes g_{j_1}) \wedge (f_{i_2} \otimes g_{j_2}) \wedge (f_{i_3} \otimes g_{j_3}) & j_4 < \cdots < j_{n+3}, \quad j_4 \leqslant j_{n+4}, \\ \otimes d_\lambda(f_{i_4} \wedge \cdots \wedge f_{i_{n+3}} \otimes f_{i_{n+4}}) & i_1 \leqslant i_2 \leqslant i_3. \\ d_\lambda(g_{j_4} \wedge \cdots \wedge g_{j_{n+3}} \otimes g_{j_{n+4}}) & \text{If } i_k = i_{k+1}, \text{ then } j_k < j_{k+1} \\ & \quad \text{for } k = 1, 2 \end{array} \right\}$$

where $i_1, ..., i_{n+4}$, $j_1, ..., j_{n+4}$ are positive integers less than or equal to $n+2$, and $f_r \# f_s$ stands for $f^{(1)}_r f^{(1)}_s$ when $r \neq s$, $f^{(2)}_r$ when $r = s$. It is easy to check that $\mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ are R-free bases of $\Lambda^{n+2} F \otimes D_2 G \otimes \Lambda^{n+2} G \otimes D_2 F$, $\Lambda^4(F \otimes G) \otimes \Lambda^n F \otimes \Lambda^n G$ and $\Lambda^3(F \otimes G) \otimes L_\lambda F \otimes L_\lambda G$, respectively. To each element contained in $\mathscr{B}_1, \mathscr{B}_2$ or $\mathscr{B}_3$, we associate $2n+4$ integers $(a_1, ..., a_{n+2}; b_1, ..., b_{n+2})$ where a_k is the number of i_t equal to k and b_k is the number of j_t equal to k. It is easy to see that this gives us gradations of the three modules, and our complex is decomposed into direct summands. We have only to show that each direct summand is exact.

For the simplicity of notations, we put $(A; B) = (a_1, ..., a_{n+2}; b_1, ..., b_{n+2})$. The symbol $(*)_{(A;B)}$ would mean the direct summand corresponding to $(A; B)$.

At first assume the set $\{a_1, ..., a_{n+2}, b_1, ..., b_{n+2}\}$ contains 0. Obviously $(\Lambda^{n+2}F \otimes D_2 G \otimes \Lambda^{n+2} G \otimes D_2 F)_{(A;B)}$ must be 0. In order to show the exactness of the direct summand corresponding to $(A; B)$, we must prove that $\partial_{(A;B)}$ is injective. With no loss of generality, we may assume that $b_{n+2} = 0$. Let H be the free submodule of G spanned by $g_1, ..., g_{n+1}$. We have only to show that the map

$$\Lambda^4(F \otimes H) \otimes \Lambda^n F \otimes \Lambda^n H \xrightarrow{\partial'} \Lambda^3(F \otimes H) \otimes L_\lambda F \otimes L_\lambda H$$

is injective, where ∂' is defined like ∂. In this case, by the Akin–Buchsbaum–Weyman complex and the Eagon–Northcott complex, we get $X_4^n(n+2, n+1) = X_3^{n+1}(n+2, n+1) = 0$. By Proposition II.6.4, we obtain $Z_4^n(n+2, n+1) = 0$. This implies that ∂' is injective.

Next assume that all a_k's and b_k's are positive integers. Let $\mathfrak{S} = \mathfrak{S}_{n+2} \times \mathfrak{S}_{n+2}$ be a subgroup of $GL(F) \times GL(G)$ such that each $\sigma = (\tau, \rho) \in \mathfrak{S}$ acts with $\tau(f_i) = f_{\tau(i)}$ and $\rho(g_j) = g_{\rho(i)}$, where $\mathfrak{S}_{n+2}$ is the symmetric group on $\{1, ..., n+2\}$. It is easy to see that $\sigma = (\tau, \rho)$ sends the direct summand corresponding to $(a_1, ..., a_{n+2}; b_1, ..., b_{n+2})$ to one corresponding to $(a_{\tau^{-1}(1)}, ..., a_{\tau^{-1}(n+2)}; b_{\rho^{-1}(1)}, ..., b_{\rho^{-1}(n+2)})$ isomorphically. Therefore we have only to show exactness when

$$\begin{aligned}(A; B) = {} & (1, ..., 1, 3; 1, ..., 1, 3),\\ & (1, ..., 1, 3; 1, ..., 1, 2, 2),\\ & (1, ..., 1, 2, 2; 1, ..., 1, 3),\\ & (1, ..., 1, 2, 2; 1, ..., 1, 2, 2).\end{aligned}$$

We prove exactness when $(A; B) = (1, ..., 1, 2, 2; 1, ..., 1, 2, 2)$. The other three cases would be proved more easily.

Suppose $(A; B) = (1, ..., 1, 2, 2; 1, ..., 1, 2, 2)$. By the definition of $(A; B)$, $(\Lambda^{n+2}F \otimes D_2 G \otimes \Lambda^{n+2} G \otimes D_2 F)_{(A;B)}$ is a free module of rank 1 generated by $f_1 \wedge \cdots \wedge f_{n+2} \otimes g_{n+1}^{(1)} \cdot g_{n+2}^{(1)} \otimes g_1 \wedge \cdots \wedge g_{n+2} \otimes f_{n+1}^{(1)} \cdot f_{n+2}^{(1)}$. Then we have

$$\begin{aligned}&\delta_{(A;B)}(f_1 \wedge \cdots \wedge f_{n+2} \otimes g_{n+1}^{(1)} \cdot g_{n+2}^{(1)} \otimes g_1 \wedge \cdots \wedge g_{n+2} \otimes f_{n+1}^{(1)} \cdot f_{n+2}^{(1)})\\ &\quad = \sum_{1 \leqslant i < j \leqslant n+2} \sum_{1 \leqslant r < t \leqslant n+2} \sum_{\substack{\{c_1, c_2\} = \{n+1, n+2\}\\ \{d_1, d_2\} = \{n+1, n+2\}}} (-1)^{i+j+r+t}\\ &\qquad \times \begin{pmatrix} i & j & d_1 & d_2 \\ c_1 & c_2 & r & t \end{pmatrix} \otimes f_1 \wedge \overset{i}{\vee} \cdots \overset{j}{\vee} \wedge f_{n+2} \otimes g_1 \wedge \overset{r}{\vee} \cdots \overset{t}{\vee} \wedge g_{n+2}\\ &\quad \neq 0,\end{aligned}$$

where $f_1 \wedge \overset{i}{\vee}\cdots\overset{j}{\vee} \wedge f_{n+2}$ means

$$f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_{n+2},$$

and $\binom{i\ j\ d_1\ d_2}{c_1\ c_2\ r\ t}$ stands for

$$(f_i \otimes g_{c_1}) \wedge (f_j \otimes g_{c_2}) \wedge (f_{d_1} \otimes g_r) \wedge (f_{d_2} \otimes g_t).$$

So $\delta_{(A;B)}$ is injective. Therefore we have only to show that $\mathrm{Im}(\delta_{(A;B)}) \supseteq \mathrm{Ker}(\partial_{(A;B)})$.

Let $\mathscr{B}'_2$, $\mathscr{B}''_2$, and $\mathscr{B}_2^{(d,e)}$ be subsets of $\mathscr{B}_2$ defined to be

$$\mathscr{B}'_2 = \{b \in \mathscr{B}_2 \mid b \in (\Lambda^4(F \otimes G) \otimes \Lambda^n F \otimes \Lambda^n G)_{(A;B)}\},$$

$$\mathscr{B}''_2 = \left\{ \begin{matrix} \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix} \\ \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \\ \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \end{matrix} \in \mathscr{B}'_2 \;\middle|\; \begin{matrix} \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix} = \varepsilon \begin{pmatrix} n+1 & n+2 & t & u \\ r & s & n+1 & n+2 \end{pmatrix} \\ \text{is satisfied for some integers} \\ r, s, t, u, \text{ and } \varepsilon = \pm 1. \end{matrix} \right\},$$

$$\mathscr{B}_2^{(d,e)} = \left\{ \begin{matrix} \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix} \\ \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \\ \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \end{matrix} \in \mathscr{B}_2 \;\middle|\; \begin{matrix} {}^{\#}\{k \mid 1 \leqslant k \leqslant 4,\, i_k \geqslant n+1.\} = d \\ \\ {}^{\#}\{k \mid 1 \leqslant k \leqslant 4,\, j_k \geqslant n+1.\} = e \end{matrix} \right\},$$

where d and e are positive integers.

Let $C = \sum_{b \in \mathscr{B}'_2} c_b \cdot b$ be a given element contained in $\mathrm{Ker}(\partial_{(A;B)})$, where c_b's are elements in the coefficient ring R. We wish to show that $C = \sum_{b \in \mathscr{B}'_2} c_b \cdot b$ is in $\mathrm{Im}(\delta_{(A;B)})$. At first we show that $c_b = 0$ when b is in $\mathscr{B}'_2 \backslash \mathscr{B}''_2$.

(0) Suppose $b_0 = \binom{i_1\ i_2\ i_3\ i_4}{j_1\ j_2\ j_3\ j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(2,2)}$.

By definition, there exists t such that $1 \leqslant t \leqslant 4$ and both i_t and j_t are contained in $\{n+1, n+2\}$. Then it is easy to see that b_0 is an only one element in $\mathscr{B}'_2$ such that, when $\partial_{(A;B)}(b_0)$ is written in the form $\sum_{h \in \mathscr{B}_3} q_h \cdot h$, $q_{h_0} \neq 0$ is satisfied, where

$$h_0 = \begin{pmatrix} i_1 \overset{t}{\vee} \cdots i_4 \\ j_1 \cdots j_4 \end{pmatrix} \otimes d_\lambda(f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes f_{i_t}) \otimes d_\lambda(g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \otimes g_{j_t}).$$

Therefore we obtain $c_{b_0} = 0$.

(1) Suppose $b_1 = \binom{i_1 i_2 i_3 i_4}{j_1 j_2 j_3 j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(3,2)}$.

By definition, there exists t such that $1 \leqslant t \leqslant 4$ and both i_t and j_t are contained in $\{n+1, n+2\}$. Let h_1 be contained in $\mathscr{B}_3$,

$$h_1 = \begin{pmatrix} i_1 \cdots \overset{t}{\vee} \cdots i_4 \\ j_1 \cdots j_4 \end{pmatrix} \otimes d_\lambda(f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes f_{i_t}) \otimes d_\lambda(g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \otimes g_{j_t}),$$

and $\mathscr{D}_1$ be a subset of $\mathscr{B}'_2$,

$$\mathscr{D}_1 = \left\{ b \in \mathscr{B}'_2 \,\middle|\, \text{when } \partial_{(A\,;\,B)}(b) = \sum_{h \in \mathscr{B}_3} q_h \cdot h,\ q_{h_1} \text{ is not } 0 \right\}.$$

Then it is easy to see that $\mathscr{D}_1 \backslash \{b_1\}$ is included in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(2,2)}$. By (0) we have already that $c_{b_0} = 0$ when b_0 is in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(2,2)}$. Therefore c_{b_1} must be equal to 0.

(2) Suppose $b_2 = \binom{i_1 i_2 i_3 i_4}{j_1 j_2 j_3 j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(2,3)}$.

By the same argument as (1), c_{b_2} is equal to 0.

(3) Suppose $b_3 = \binom{i_1 i_2 i_3 i_4}{j_1 j_2 j_3 j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(3,3)}$.

By definition, there exists t such that $1 \leqslant t \leqslant 4$ and both i_t and j_t are contained in $\{n+1, n+2\}$. Let h_3 be contained in $\mathscr{B}_3$,

$$h_3 = \begin{pmatrix} i_1 \cdots \overset{t}{\vee} \cdots i_4 \\ j_1 \cdots j_4 \end{pmatrix} \otimes d_\lambda(f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes f_{i_t}) \otimes d_\lambda(g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \otimes g_{j_t}),$$

and $\mathscr{D}_3$ be a subset of $\mathscr{B}'_2$,

$$\mathscr{D}_3 = \left\{ b \in \mathscr{B}'_2 \,\middle|\, \text{when } \partial_{(A\,;\,B)}(b) = \sum_{h \in \mathscr{B}_3} q_h \cdot h,\ q_{h_3} \text{ is not } 0 \right\}.$$

Then it is easy to see that $\mathscr{D}_3 \backslash \{b_3\}$ is included in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap (\mathscr{B}_2^{(2,2)} \cup \mathscr{B}_2^{(3,2)} \cup \mathscr{B}_2^{(2,3)})$. By (0), (1), and (2) we obtain $c_{b_3} = 0$.

(4) Suppose $b_4 = \binom{i_1 i_2 i_3 i_4}{j_1 j_2 j_3 j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(4,2)}$.

By definition, there exists t such that $1 \leqslant t \leqslant 4$ and both i_t and j_t are contained in $\{n+1, n+2\}$. Let h_4 be contained in $\mathscr{B}_4$,

$$h_4 = \begin{pmatrix} i_1 \cdots \overset{t}{\vee} \cdots i_4 \\ j_1 \cdots j_4 \end{pmatrix} \otimes d_\lambda(f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes f_{i_t}) \otimes d_\lambda(g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \otimes g_{j_t}),$$

and $\mathscr{D}_4$ be a subset of $\mathscr{B}'_2$,

$$\mathscr{D}_4 = \left\{ b \in \mathscr{B}'_2 \,\middle|\, \text{when } \partial_{(A;B)}(b) = \sum_{h \in \mathscr{B}_3} q_h \cdot h,\ q_{h_4} \text{ is not } 0 \right\}.$$

Then it is easy to see that $\mathscr{D}_4 \backslash \{b_4\}$ is included in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(3,2)}$. By (2) c_{b_4} is equal to 0.

(5) Suppose $b_5 = \binom{i_1\, i_2\, i_3\, i_4}{j_1\, j_2\, j_3\, j_4} \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(2,4)}$.

By the same argument as for (4), we can show that c_{b_5} is equal to 0.

(6) It is easy to show that

$$\begin{aligned}(\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(4,3)} &= (\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(3,4)} \\ &= (\mathscr{B}'_2 \backslash \mathscr{B}''_2) \cap \mathscr{B}_2^{(4,4)} = \varnothing.\end{aligned}$$

By (1), ..., (6), we may assume that $C = \sum_{b \in \mathscr{B}'_2} c_b \cdot b$. Therefore C is written in the form

$$\begin{aligned}
&\sum_{1 \leqslant i<j \leqslant n+2} \sum_{1 \leqslant k<t \leqslant n+2} (-1)^{i+j+k+t} \\
&\quad \times A^{ij}_{kt} \begin{pmatrix} i & j & n+1 & n+2 \\ n+1 & n+2 & k & t \end{pmatrix} \otimes [i,j;k,t] \\
&\quad + \sum_{1 \leqslant j<i \leqslant n+2} \sum_{1 \leqslant k<t \leqslant n+2} (-1)^{i+j+k+t+1} \\
&\quad \times A^{ij}_{kt} \begin{pmatrix} i & j & n+1 & n+2 \\ n+1 & n+2 & k & t \end{pmatrix} \otimes [j,i;k,t] \\
&\quad + \sum_{1 \leqslant i<j \leqslant n+2} \sum_{1 \leqslant t<k \leqslant n+2} (-1)^{i+j+k+t+1} \\
&\quad \times A^{ij}_{kt} \begin{pmatrix} i & j & n+1 & n+2 \\ n+1 & n+2 & k & t \end{pmatrix} \otimes [i,j;t,k] \\
&\quad + \sum_{1 \leqslant j<i \leqslant n+2} \sum_{1 \leqslant t<k \leqslant n+2} (-1)^{i+j+k+t} \\
&\quad \times A^{ij}_{kt} \begin{pmatrix} i & j & n+1 & n+2 \\ n+1 & n+2 & k & t \end{pmatrix} \otimes [j,i;t,k],
\end{aligned}$$

where A^{ij}_{kt}'s are elements in the coefficient ring R, and $[i,j;k,t]$ means $f_1 \wedge \overset{\overset{i\ j}{\vee\ \vee}}{\cdots} \wedge f_{n+2} \otimes g_1 \wedge \overset{\overset{k\ t}{\vee\ \vee}}{\cdots} \wedge g_{n+2}$.

By direct computation we know that A^{ij}_{kt} must be equal to A^{12}_{12} when

$i \neq j$, $k \neq t$, $i \leqslant n$ and $j \leqslant n$. Similarly A_{kt}^{ij} must be equal to A_{12}^{12} when $i \neq j$, $k \neq t$, $k \leqslant n$, and $t \leqslant n$. Replace C by

$$C - A_{12}^{12} \cdot \delta_{(A;B)}(f_1 \wedge \cdots \wedge f_{n+2} \otimes g_{n+1}^{(1)} \cdot g_{n+2}^{(1)} \otimes g_1 \wedge \cdots \wedge g_{n+2} \otimes f_{n+1}^{(1)} \cdot f_{n+2}^{(1)}).$$

Then A_{kt}^{ij} is equal to 0 when $\max\{i, j\}$ or $\max\{k, t\}$ is less than or equal to n.

We must show that $C = 0$. By the definition of C, it is contained in the free submodule spanned by $\mathscr{B}_2'' \cap (\mathscr{B}_2^{(3,3)} \cup \mathscr{B}_2^{(4,3)} \cup \mathscr{B}_2^{(3,4)} \cup \mathscr{B}_2^{(4,4)})$. Therefore C is written in the form

$$C = \sum_{b_1 \in \mathscr{B}_2'' \cap \mathscr{B}_2^{(3,3)}} c_{b_1} \cdot b_1 + \sum_{b_2 \in \mathscr{B}_2'' \cap \mathscr{B}_2^{(4,3)}} c_{b_2} \cdot b_2$$
$$+ \sum_{b_3 \in \mathscr{B}_2'' \cap \mathscr{B}_2^{(3,4)}} c_{b_3} \cdot b_3 + \sum_{b_4 \in \mathscr{B}_2'' \cap \mathscr{B}_2^{(4,4)}} c_{b_4} \cdot b_4.$$

Suppose $b_1 = \left(\begin{smallmatrix} i_1 i_2 i_3 i_4 \\ j_1 j_2 j_3 j_4 \end{smallmatrix}\right) \otimes f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes g_{j_5} \wedge \cdots \wedge g_{j_{n+4}}$ is contained in $\mathscr{B}_2'' \cap \mathscr{B}_2^{(3,3)}$.

By definition there exists r such that $1 \leqslant r \leqslant 4$ and both i_r and j_r are contained in $\{n+1, n+2\}$. Let h_1 be contained in $\mathscr{B}_3$,

$$\begin{pmatrix} i_1 \cdots i_{r-1}\, i_{r+1} \cdots i_4 \\ j_1 \cdots j_{r-1}\, j_{r+1} \cdots j_4 \end{pmatrix} \otimes d_\lambda(f_{i_5} \wedge \cdots \wedge f_{i_{n+4}} \otimes f_{i_r})$$
$$\otimes d_\lambda(g_{j_5} \wedge \cdots \wedge g_{j_{n+4}} \otimes g_{j_r}),$$

and $\mathfrak{F}_1$ be a subset of $\mathscr{B}_2''$,

$$\mathfrak{F}_3 = \left\{ b \in \mathscr{B}_2'' \,\middle|\, \text{when } \partial_{(A;B)}(b) = \sum_{h \in \mathscr{B}_3} q_h \cdot h,\, q_{h_1} \text{ is not } 0 \right\}.$$

Then it is easy to see that $\mathfrak{F} \backslash \{b_1\}$ is included in $\mathscr{B}_2^{(2,2)} \cup \mathscr{B}_2^{(2,3)} \cup \mathscr{B}_2^{(3,2)}$. By the definition of C, c_{b_1} must be equal to 0.

By the same arguments as above, we can prove that $C = 0$.

We have completed the proof of Lemma V.6.1. Q.E.D.

Consequently $X_5^n(n+2, n+2)$ is proved to be universally free.

2. *In the Case Where $n \neq 4$*

In this section we prove the existence of the minimal free resolutions of determinantal ideals defined by n-minors of the generic $(n+2)$-square matrices in the case where $n \neq 4$.

Let R be an arbitrary prime field k.

DEFINITION V.2.1. When $\mathrm{ch}(k)=p$ (p is 0 or a prime integer), $\beta_p(i, n)$ is defined to be

$$\beta_p(i, n)=\dim_k \mathrm{Tor}_i^S(S/I_n, S/I_1).$$

PROPOSITION V.2.2. *Over an arbitrary prime field k,*

$$\underline{\mathrm{Ext}}_S^i(S/I_n, S)=\begin{cases}(S/I_n)(3n+6) & \text{when } i=9\\ 0 & \text{otherwise.}\end{cases}$$

Proof. When k is the rationals $\mathbb{Q}$, it is easily checked by Theorem II.1.1, Theorem II.1.5, and Lascoux's resolutions [17].

By Theorem II.1.1, for $i\neq 9$, $\underline{\mathrm{Ext}}_S^i(S/I_n, S)$ is equal to 0 over an arbitrary prime field k. Moreover by Proposition II.2.4, we have

$$\underline{\mathrm{Ext}}_S^9(S/I_n, S)=K_{(S/I_n)}.$$

It is well known that $K_S=S(-(n+2)^2)$ (cf. [11, Corollary (2.2.8)]). Hence we have

$$\underline{\mathrm{Ext}}_S^9(S/I_n, S)=\underline{\mathrm{Ext}}_S^9(S/I_n, K_S)((n+2)^2)=K_{(S/I_n)}((n+2)^2).$$

By Remark II.2.6, there exists ρ such that

$$K_{(S/I_n)}=(S/I_n)(\rho)$$

and ρ does not depend on $\mathrm{ch}(k)$. Therefore we have

$$\underline{\mathrm{Ext}}_S^9(S/I_n, S)=K_{(S/I_n)}((n+2)^2)=(S/I_n)(\rho+(n+2)^2),$$

where $\rho+(n+2)^2$ does not depend on $\mathrm{ch}(k)$. Therefore it must be equal to $3n+6$. Q.E.D.

Let $\mathbb{F}(p, n)$. be a minimal free resolution of S/I_n over the prime field of characteristic p. By definition V.2.1, $\mathrm{rank}_S\, \mathbb{F}(p, n)_i$ is equal to $\beta_p(i, n)$. Since $\underline{\mathrm{Hom}}_S(\mathbb{F}(p, n), S)(-3n-6)$ is also a minimal free resolution of S/I_n by Proposition V.2.2, we obtain $\beta_p(0, n)=\beta_p(9, n)$, $\beta_p(1, n)=\beta_p(8, n)$, $\beta_p(2, n)=\beta_p(7, n)$, $\beta_p(3, n)=\beta_p(6, n)$ and $\beta_p(4, n)=\beta_p(5, n)$ for any p and n. Therefore in order to show the existence of the minimal free resolution over the integers $\mathbb{Z}$, we have only to show that $\beta_p(0, n)$, $\beta_p(1, n)$, $\beta_p(2, n)$, $\beta_p(3, n)$, and $\beta_p(4, n)$ do not depend on p (cf. Proposition II.3.4).

We have already seen that $\beta_p(0, n)$, $\beta_p(1, n)$, $\beta_p(2, n)$, and $\beta_p(3, n)$ do not depend on p when $n\geqslant 2$ (see Section 2 of Chapter IV).

In the rest of this section we show that $\beta_p(4, n)$ does not depend on p when $n\neq 4$.

DEFINITION V.2.3. Since $F(p, n)_i$ is a graded free module for each p, n, and i, there exist $\beta_p(i, n)_t$'s such that

$$F(p, n)_i = \bigoplus_t S(-t)^{\beta_p(i,n)_t}, \left(\beta_p(i, n) = \sum_t \beta_p(i, n)_t\right).$$

It is easy to check that $\beta_p(i, n)_t$'s do not depend on the choice of a minimal free resolution $\mathbb{F}(p, n)$..

Remark V.2.4. When the coefficient field k is equal to $\mathbb{Q}$, the minimal free resolution $\mathbb{F}(0, n)$. is written in the form

$$F(0, n)_0 = S, \qquad F(0, n)_1 = S(-n)^{\beta_0(1,n)_n},$$
$$F(0, n)_2 = S(-n-1)^{\beta_0(2,n)_{n+1}}, \qquad F(0, n)_3 = S(-n-2)^{\beta_0(3,n)_{n+2}},$$
$$F(0, n)_4 = S(-n-3)^{\beta_0(4,n)_{n+3}} \oplus S(-2n-2)^{\beta_0(4,n)_{2n+2}},$$
$$F(0, n)_5 = S(-n-4)^{\beta_0(5,n)_{n+4}} \oplus S(-2n-3)^{\beta_0(5,n)_{2n+3}},$$
$$F(0, n)_6 = S(-2n-4)^{\beta_0(6,n)_{2n+4}}, \qquad F(0, n)_7 = S(-2n-5)^{\beta_0(7,n)_{2n+5}},$$
$$F(0, n)_8 = S(-2n-6)^{\beta_0(8,n)_{2n+6}}, \qquad F(0, n)_9 = S(-3n-6),$$

by Lascoux's complex.

To a minimal free resolution $\mathbb{F}(p, n)$., we associate a figure where "$\circ$" on (i, j) means $\beta_p(i, n)_j \neq 0$ and no mark on (i, j) means $\beta_p(i, n)_j = 0$. For example, to a minimal free resolution $\mathbb{F}(0, n)$. over $\mathbb{Q}$, we can associate Fig. 1 by Definition V.2.3 and Remark V.2.4.

Now assume that $\mathrm{ch}(k) = p > 0$. Since $\mathbb{F}(p, n)$. is a minimal free resolution, $\underline{\mathrm{Hom}}_S(\mathbb{F}(p, n).,\, S)(-3n-6)$ is also a minimal free resolution of S/I_n. Therefore $F(p, n)_9 = S(-3n-6)$, $F(p, n)_8 = S(-2n-6)^{\beta_p(8,n)_{2n+6}}$, $F(p, n)_7 = S(-2n-5)^{\beta_p(7,n)_{2n+5}}$, $F(p, n)_6 = S(-2n-4)^{\beta_p(6,n)_{2n+4}}$ are satisfied

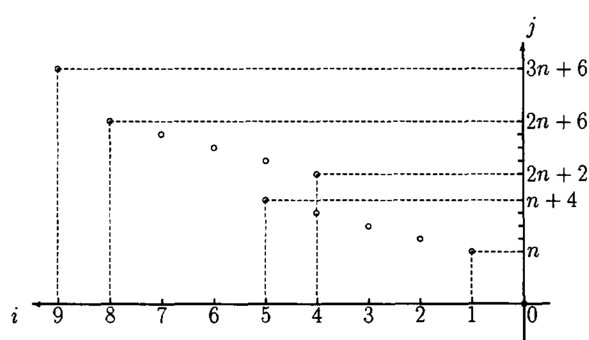

FIGURE 1

and their ranks do not depend on p. By the minimality of $\mathbb{F}(p, n)$. and $\underline{\mathrm{Hom}}_S(\mathbb{F}(p, n)., S)(-3n-6)$, we have

$$\beta_p(4, n)_j = 0 \qquad \text{for} \quad j < n+3 \text{ or } j > 2n+2,$$

$$\beta_p(5, n)_j = 0 \qquad \text{for} \quad j < n+4 \text{ or } j > 2n+3.$$

Then Fig. 2 is obtained over an arbitrary field k. By the same argument as above, $\beta_p(i, n)$ does not depend on p when $i \neq 4, 5$. Further by Section 1 of this chapter, $\beta_p(5, n)_{n+4}$ and $\beta_p(4, n)_{2n+2}$ do not depend on p. Consequently, $\beta_p(i, n)_j$'s do not depend on p for all (i, j) in the plane except for $\mathbf{Q}$. By comparing Fig. 1 with Remark V.2.4, we have to show that $\beta_p(i, n)_j = 0$ when (i, j) is in $\mathbf{Q}$, i.e.,

$$\beta_p(4, n)_{n+4} = \cdots = \beta_p(4, n)_{2n+1} = \beta_p(5, n)_{n+5} = \cdots = \beta_p(5, n)_{2n+2} = 0.$$

Remark V.2.5. By Fig.2, $\beta_p(6, n)_{n+5}$, $\beta_p(7, n)_{n+6}, \ldots$ are equal to 0 over an arbitrary prime field k. This implies that $X_6^n(n+2, n+2) = X_7^n(n+2, n+2) = \cdots = 0$ over any coefficient ring when $n \geqslant 2$.

At first, suppose $n = 2$. Then $\mathbf{Q}$ is empty. Hence in this case, there exists a minimal free resolution.

Next, suppose $n = 3$. Then $\mathbf{Q}$ consists of $(4, 7)$ and $(5, 8)$. Compare Fig. 3 with Fig. 1, and look at the degree 7 component of the minimal free resolution. Then it is easy to see that $\beta_p(4, n)_7$ must be equal to 0. By duality, so is $\beta_p(5, n)_8$. Hence there exists a minimal free resolution in this case, too.

Suppose $n \geqslant 5$. By Proposition IV.3.2, we have $\beta_p(4, n)_{n+4} = \cdots = \beta_p(4, n)_{2n} = 0$. See Fig. 4. By duality, we obtain $\beta_p(5, n)_{n+6} = \cdots = \beta_p(5, n)_{2n+2} = 0$. Since $n+5 \leqslant 2n$, both $\beta_p(4, n)_{n+4}$ and $\beta_p(4, n)_{n+5}$ are equal to 0. By the same argument as in the case where $n = 3$, one can prove that $\beta_p(5, n)_{n+5} = 0$. Then by duality, $\beta_p(4, n)_{2n+1} = 0$ is satisfied.

Consequently, we get the following theorem;

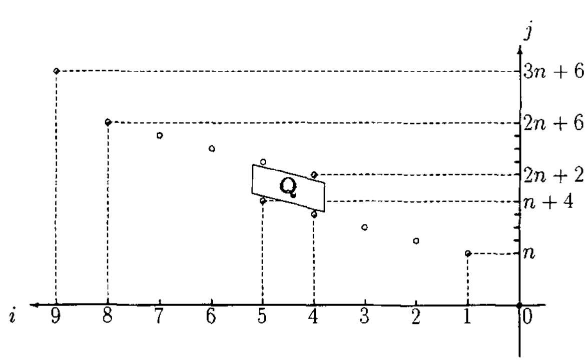

FIGURE 2

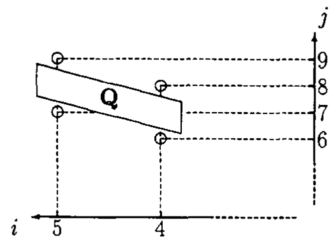

FIGURE 3

THEOREM V.2.6. *When n is not equal to 4, there exists a minimal free resolutions of S/I_n over any coefficient ring.*

3. *In the Case of* $n=4$

In the previous section, $\beta_p(0,4)$, $\beta_p(1,4)$, $\beta_p(2,4)$ and $\beta_p(3,4)$ are shown to be independent of p. In the rest of this paper we shall prove that $\beta_p(4,4)$ does not depend on p. By Proposition II.3.4, it is sufficient to show that $\mathrm{Tor}_4^S(S/I_4, S/I_1)$ is $\mathbb{Z}$-free when our coefficient ring R is the integers $\mathbb{Z}$.

Let $X=(x_{ij})$ be the generic 6×6-matrix, and $Y=(y_{kr})$ be the generic 7×7-matrix. We denote the polynomial rings $\mathbb{Z}[x_{ij}]_{1\leqslant i,j\leqslant 6}$ and $\mathbb{Z}[y_{kr}]_{1\leqslant k,r\leqslant 7}$ by S and T, respectively. By sending the matrix Y to $\left(\begin{smallmatrix} X & 0\\ 0 & 1\end{smallmatrix}\right)$, S has the T-algebra structure. Let J_5 be the ideal of T generated by 5-minors of Y, and I_4 the ideal of S generated by 4-minors of X.

By Theorem V.2.6, T/J_5 has a minimal free resolution $\mathbb{P}.$. By [17], it looks like the total complex of the double complex in Fig. 5, where the vertical boundary maps have degree 5 and horizontal degree 1. Here $X.$ must be equal to the linear complex $X_.^5(7,7)$ (cf. Remark II.6.5)

It is easy to see that $J_5S=I_4$. Since $\operatorname{depth}(J_5,S)$ is equal to $\operatorname{depth}(I_4,S)=9$, the complex $\mathbb{P}.\otimes_T S$ is acyclic from the depth sensitivity (cf. Remark II.4.2). Therefore $\mathbb{P}.\otimes_T S$ is a finite free resolution of $(T/J_5)\otimes S=S/I_4$.

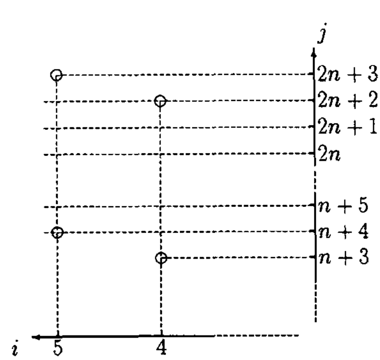

FIGURE 4

$$\begin{tikzcd}
S \arrow[d] & & & & & & \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] \arrow[d] & Y_4 \arrow[r] \arrow[d] & Y_5 \arrow[d] & & \\
& & X_5 \arrow[r] & X_4 \arrow[r] & X_3 \arrow[r] & X_2 \arrow[r] & X_1 \arrow[d] \\
& & & & & & S
\end{tikzcd}$$

FIGURE 5

By using this free resolution, we compute $\operatorname{Tor}_4^S(S/I_4, S/I_1)$.

LEMMA V.3.1. *For any commutative ring R, $H_5((X.\otimes_T (S/I_1))\otimes_{\mathbb{Z}} R)$ is an R-free module and its rank does not depend on R.*

Proof. We have only to prove this lemma when R is a field k.

By Proposition II.6.4, there exists an exact sequence $\cdots \to H_5((X.^6(7,7)\otimes_T (S/I_1))\otimes_{\mathbb{Z}} k) \to H_5((X.^5(7,7)\otimes_T (S/I_1))\otimes_{\mathbb{Z}} k) \to H_5((Z.^5(7,7)\otimes_T (S/I_1))\otimes_{\mathbb{Z}} k) \to H_4((X.^6(7,7)\otimes_T (S/I_1))\otimes_{\mathbb{Z}} k) \to \cdots$. Since $X_4^6(7,7)=X_5^6(7,7)=0$ by the Gulliksen-Negård complex [13], it is sufficient to show that $\dim_k(H_5(Z.^5(7,7)\otimes_T C))$ does not depend on k, where C is the T-algebra $(S/I_1)\otimes_{\mathbb{Z}} k$. Recall that by Lemma V.1.7, $Z_5^5(7,7)\otimes_T C$ coincides with $\Lambda^7 F\otimes D_2 G\otimes_C \Lambda^7 G\otimes_C D_2 F$, where both F and G are C-free modules of rank 7. Let $\{f_1, ..., f_7\}$ and $\{g_1, ..., g_7\}$ be free bases of F and G, respectively.

We define the C-free modules as

$$U_5^5=\Lambda^7 F\otimes D_2 G\otimes \Lambda^7 G\otimes D_2 F,$$

$$U_5^5=(\Lambda^6 F\otimes G\otimes \Lambda^7 G\otimes D_2 F)\oplus(\Lambda^7 F\otimes D_2 G\otimes \Lambda^6 G\otimes F),$$

$$U_3^6=\Lambda^7 F\otimes G\otimes \Lambda^7 G\otimes F,$$

$$V_5=\Lambda^4(F\otimes G)\otimes \Lambda^5 F\otimes \Lambda^5 G,$$

$$V_4=\Lambda^3(F\otimes G)\otimes \Lambda^5 F\otimes \Lambda^5 G,$$

where all tensor products are defined over C. Consider the diagram

$$\begin{tikzcd}
& U_5^5 \arrow[r, "\delta"] \arrow[d, "\zeta"] & V_5 \arrow[d, "\rho"] \\
U_3^6 \arrow[r, "\xi"] & U_4^5 \arrow[r, "\delta'"] & V_4
\end{tikzcd}$$

where δ, δ', ξ, ζ, and ρ are defined as follows:

$\delta: U_5^5 \to V_5$ is the map defined in Lemma V.1.7.

$\rho: V_5 \to V_4$ is the Koszul map, that is, it is the composite map

$$\Lambda^4(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G \xrightarrow{\Delta\otimes 1} (F\otimes G)\otimes\Lambda^3(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G$$
$$\xrightarrow{\text{ev}\otimes 1} \Lambda^3(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G,$$

where $\text{ev}: F\otimes G \to C$ is the map between C-free modules sending the 7×7-matrix $(f_i\otimes g_j)$ to

$$\begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

ζ is the sum of the two composite maps

$$\Lambda^7F\otimes D_2G\otimes\Lambda^7G\otimes D_2F \xrightarrow{\Delta\otimes\Delta\otimes 1\otimes 1} F\otimes\Lambda^6F\otimes G\otimes G\otimes\Lambda^7G\otimes D_2F$$
$$\xrightarrow{T} (F\otimes G)\otimes(\Lambda^6F\otimes G\otimes\Lambda^7G\otimes D_2G)$$
$$\xrightarrow{\text{ev}\otimes 1} \Lambda^6F\otimes G\otimes\Lambda^7G\otimes D_2F$$

and

$$\Lambda^7F\otimes D_2G\otimes\Lambda^7G\otimes D_2F \xrightarrow{1\otimes 1\otimes\Delta\otimes\Delta} \Lambda^7F\otimes D_2G\otimes G\otimes\Lambda^6G\otimes F\otimes F$$
$$\xrightarrow{T} (F\otimes G)\otimes(\Lambda^7F\otimes G\otimes\Lambda^6G\otimes D_2G)$$
$$\xrightarrow{\text{ev}\otimes 1} \Lambda^7F\otimes G\otimes\Lambda^6G\otimes D_2F.$$

ξ is the sum of the two composite maps

$$\Lambda^7F\otimes G\otimes\Lambda^7G\otimes F \xrightarrow{1\otimes 1\otimes\Delta\otimes 1} \Lambda^7F\otimes G\otimes G\otimes\Lambda^6G\otimes F$$
$$\xrightarrow{1\otimes m\otimes 1\otimes 1} \Lambda^7F\otimes D_2G\otimes\Lambda^6G\otimes F$$

and

$$\Lambda^7F\otimes G\otimes\Lambda^7G\otimes F \xrightarrow{\Delta\otimes 1\otimes 1\otimes 1} F\otimes\Lambda^6F\otimes G\otimes\Lambda^7G\otimes F$$
$$\xrightarrow{T} \Lambda^6F\otimes G\otimes\Lambda^7G\otimes F\otimes F$$
$$\xrightarrow{1\otimes m\otimes 1\otimes 1} \Lambda^7F\otimes D_2G\otimes\Lambda^6G\otimes F.$$

δ' is the sum of the two maps

$$\begin{aligned}
\Lambda^6F\otimes G\otimes\Lambda^7G\otimes D_2F &\xrightarrow{\Lambda 1\times\Lambda\otimes 1} F\otimes\Lambda^5F\otimes G\otimes\Lambda^2G\otimes\Lambda^5G\otimes D_2F\\
&\xrightarrow{T} (F\otimes G)\otimes(D_2F\otimes\Lambda^2G)\otimes(\Lambda^5F\otimes\Lambda^5G)\\
&\xrightarrow{1\otimes\theta\otimes 1} (F\otimes G)\otimes\Lambda^2(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G\\
&\xrightarrow{m\otimes 1} \Lambda^3(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G
\end{aligned}$$

and

$$\begin{aligned}
\Lambda^7F\otimes D_2G\otimes\Lambda^6G\otimes F &\xrightarrow{\Lambda\otimes 1\otimes\Lambda\otimes 1} \Lambda^2F\otimes\Lambda^5F\otimes D_2G\otimes G\otimes\Lambda^5G\otimes F\\
&\xrightarrow{T} (D_2F\otimes\Lambda^2G)\otimes(F\otimes G)\otimes(\Lambda^5F\otimes\Lambda^5G)\\
&\xrightarrow{1\otimes\theta\otimes 1} \Lambda^2(F\otimes G)\otimes(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G\\
&\xrightarrow{m\otimes 1} \Lambda^3(F\otimes G)\otimes\Lambda^5F\otimes\Lambda^5G.
\end{aligned}$$

It is easy to check that $\rho\circ\delta=\delta'\circ\zeta$.

Now we show that $0\to U_3^6\xrightarrow{\xi}U_4^5\xrightarrow{\delta'}V_4$ is exact. Define $(A;B)=(a_1,...,a_7;b_1,...,b_7)$ as in the proof of Lemma V.1.7. When the set $\{a_1,...,a_7,b_1,...,b_7\}$ contains 0, $(U_3^6)_{(A;B)}=0$. In this case, $\delta'_{(A;B)}$ is obviously injective. Therefore we may assume $(A;B)=(2,1,...,1;2,1,...,1)$. By direct computations, it is easy to check that $\xi_{(A;B)}$ is injective and $\delta'_{(A;B)}\circ\xi_{(A;B)}$ is equal to 0. Hence, $\mathrm{Im}(\delta'_{(A;B)})$ is included in $\mathrm{Ker}(\xi_{(A;B)})$. So we have only to show the opposite inclusion. Assume that $\mathrm{Ker}(\delta'_{(A;B)})$ contains the element

$$\begin{aligned}
K=&\,p_1\cdot f_2\wedge\cdots\wedge f_7\otimes g_1\otimes g_1\wedge\cdots\wedge g_7\otimes f_1^{(2)}\\
&+\sum_{i=2}^{7}p_i\cdot f_1\wedge\overset{i}{\check{\cdots}}\wedge f_7\otimes g_1\otimes g_1\wedge\cdots\wedge g_7\otimes f_1f_i\\
&+q_1\cdot f_1\wedge\cdots\wedge f_7\otimes g_1^{(2)}\otimes g_2\wedge\cdots\wedge g_7\otimes f_1\\
&+\sum_{j=2}^{7}q_i\cdot f_1\wedge\cdots\wedge f_7\otimes g_1g_j\otimes g_1\wedge\overset{j}{\check{\cdots}}\wedge g_7\otimes f_1,
\end{aligned}$$

where p_i's and q_j's are elements in C. Since $\delta'_{(A;B)}(K)$ is equal to 0, we obtain $(-1)^ip_i=(-1)^jq_j$ for $i\geqslant 2$ and $j\geqslant 2$. Replace K by $K+\xi_{(A;B)}(f_1\wedge\cdots\wedge f_7\otimes g_1\otimes g_1\wedge\cdots\wedge g_7\otimes f_1)$. Then we may assume $p_i=q_j=0$ for $i\geqslant 2$ and $j\geqslant 2$. In this case it is easy to see that $p_1=q_1=0$ is satisfied. So $\mathrm{Ker}(\xi_{(A;B)})$ is included in $\mathrm{Im}(\delta'_{(A;B)})$. Consequently $0\to U_3^6\xrightarrow{\xi}U_4^5\xrightarrow{\delta'}V_4$ is proved to be exact.

By the exactness of $0 \to U_3^6 \xrightarrow{\xi} U_4^5 \xrightarrow{\delta'} V_4$, we have

$$\begin{aligned}
&\operatorname{Ker}((X_5 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k \to (X_4 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k) \\
&= H_5(Z.^5(7,7) \otimes_T C) \\
&= \operatorname{Ker}(\rho \circ \delta) \\
&= \operatorname{Ker}(\delta' \circ \zeta) \\
&= \xi^{-1}(\operatorname{Im}(\xi)).
\end{aligned}$$

Because $\operatorname{Im}(\zeta) \cap \operatorname{Im}(\xi) = 0$, $\zeta^{-1}(\operatorname{Im}(\xi)) = \operatorname{Ker}(\zeta)$ is satisfied. Therefore to prove this lemma, It suffices to show that $\dim_k(\operatorname{Ker}(\zeta))$ does not depend on k.

Denote $f_1 \wedge \cdots \wedge f_7 \otimes g_i \# g_j \otimes g_1 \wedge \cdots \wedge g_7 \otimes f_s \# f_t$ by simply $\langle i, j; s, t \rangle$. Then the set $\{\langle i, j; s, t \rangle \mid 1 \leqslant i \leqslant j \leqslant 7, 1 \leqslant s \leqslant t \leqslant 7\}$ becomes a free basis of $\Lambda^7 F \otimes D_2 G \otimes \Lambda^7 G \otimes D_2 F$. Define

$$\begin{aligned}
A' &= (1, 1, 1, 1, 1, 1, 0) + \varepsilon_i + \varepsilon_j \\
B' &= (1, 1, 1, 1, 1, 1, 0) + \varepsilon_s + \varepsilon_t,
\end{aligned}$$

where $\varepsilon_1 = (\delta_{i1}, \delta_{i2}, ..., \delta_{i7})$ (δ_{kr} is a Kronecker's delta). Then obviously $\zeta(\langle i, j; s, t \rangle)$ is in $(U_4^5)_{(A'; B')}$ Therefore we have

$$\begin{aligned}
\dim_k(\ker(\zeta)) &= {}^{\#}\{\langle i, j; s, t \rangle \mid 1 \leqslant i \leqslant j \leqslant 7, 1 \leqslant s \leqslant t \leqslant 7, \zeta(\langle i, j; s, t \rangle) = 0\} \\
&= {}^{\#}\{\langle i, j; s, t \rangle \mid 1 \leqslant i \leqslant j \leqslant 6, 1 \leqslant i \leqslant j \leqslant 6\} \\
&= 21^2.
\end{aligned}$$

So it does not depend on k. We have completed the proof of Lemma V.3.1.

Q.E.D.

Remark V.3.2. It is easy to check that

$$H_i((X. \otimes_T (S/I_1)) \otimes R) = \operatorname{Tor}_i^{S \otimes_{\mathbb{Z}} R}((S/I_4) \otimes_{\mathbb{Z}} R, S/I_1)$$

for $i = 1, 2, 3$. For $i = 1, 2, 3$, $H_i((X. \otimes_Y (S/I_1)) \otimes_{\mathbb{Z}} R)$ is a finite free module whose rank does not depend on R because $\beta_i(p, 4)$ does not depend on p. Further by the previous lemma, $H_5((X. \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} R)$ is a finite free module whose rank does not depend on R. Therefore so is $H_4((X. \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} R)$.

Suppose $t = \operatorname{rank}_{\mathbb{Z}}(H_5(X. \otimes_T (S/I_1)))$ and $s = \operatorname{rank}_{\mathbb{Z}}(H_4(X. \otimes_T (S/I_1)))$. By Lemma V.3.1 and Remark V.3.3, they are both $\mathbb{Z}$-free modules.

LEMMA V.3.3. $\operatorname{Coker}(Y_4 \otimes_T (S/I_1) \to Y_5 \otimes_T (S/I_1)) = \mathbb{Z}^t$.

Proof. It is sufficient to show that

$$\mathrm{Coker}((Y_4 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k \to (Y_5 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k) = k^t$$

for any field k. We shall denote $* \otimes_{\mathbb{Z}} k$ by $\tilde{*}$. Since $\tilde{\mathbb{P}}.$ is a minimal free resolution of $\tilde{T}/J_5\tilde{T}$, $\mathrm{Hom}_{\tilde{T}}(\tilde{Y}., \tilde{T})$ is equal to $\tilde{X}.$. Therefore $\tilde{Y}. \otimes_{\tilde{T}} (\tilde{S}/I_1\tilde{S})$ coincides with $\mathrm{Hom}_{\tilde{T}}(\tilde{X}., \tilde{T}) \otimes_{\tilde{T}} (\tilde{S}/I_1\tilde{S}) = \mathrm{Hom}_k(\tilde{X}. \otimes_{\tilde{T}} (\tilde{S}/I_1\tilde{S}), k)$. So we have

$$\begin{aligned}
&\mathrm{Coker}((Y_4 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k \to (Y_5 \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k) \\
&\quad = H^5((Y. \otimes_T (S/I_1)) \otimes_{\mathbb{Z}} k) \\
&\quad = H^5((\tilde{Y}. \otimes_T (\tilde{S}/I_1\tilde{S})) \\
&\quad = \mathrm{Hom}_k(H_5(\tilde{X}. \otimes_{\tilde{T}} (\tilde{S}/I_1\tilde{S})), k) = k^t.
\end{aligned}$$

Q.E.D.

Lemma V.3.4. $\dim_{\mathbb{Q}}(\mathrm{Tor}_4^{S \otimes \mathbb{Q}}((S/I_4) \otimes \mathbb{Q}, (S/I_1) \otimes \mathbb{Q})) = t + s.$

Proof. Let $\mathbb{R}.$ be the Lascoux's resolution [23, 17, 22] of $(T \otimes \mathbb{Q})/J_5(T \otimes Q)$. $\mathbb{R}.$ is written in the form of Fig. 5.

Consider the complex $\mathbb{R}. \otimes_{T \otimes \mathbb{Q}} (S \otimes \mathbb{Q})$. It is easy to check that the vertical maps in $\mathbb{R}. \otimes_{T \otimes \mathbb{Q}} (S \otimes \mathbb{Q})$ vanish. Therefore we obtain $\dim_{\mathbb{Q}}(\mathrm{Tor}_4^{S \otimes \mathbb{Q}}((S/I_4) \otimes \mathbb{Q}, (S/I_1) \otimes \mathbb{Q})) = t + s$ by Remark V.3.2 and Lemma V.3.3. Q.E.D.

Proposition V.3.5. $\mathrm{Tor}_4^S(S/I_4, S/I_1) = \mathbb{Z}^{t+s}$.

Proof. Consider the following double complex;

$$\begin{tikzcd}
T \otimes (S/I_1) \arrow[d] & & & & & \\
Y_1 \otimes (S/I_1) \arrow[r] & \cdot \arrow[r] & Y_3 \otimes (S/I_1) \arrow[r] \arrow[d] & Y_4 \otimes (S/I_1) \arrow[r] \arrow[d] & Y_5 \otimes (S/I_1) \arrow[d] & & \\
& & X_5 \otimes (S/I_1) \arrow[r] & X_4 \otimes (S/I_1) \arrow[r] & X_3 \otimes (S/I_1) \arrow[r] & \cdot \arrow[r] & X_1 \otimes (S/I_1) \arrow[d] \\
& & & & & & T \otimes (S/I_1).
\end{tikzcd}$$

Define A and B as follows;

$$A = \mathrm{Coker}(H^4(Y. \otimes (S/I_1)) \to H_4(X. \otimes S/I_1))),$$
$$B = \mathrm{Ker}(H^5(Y. \otimes (S/I_1)) \to H_3(X. \otimes (S/I_1))).$$

By the argument of spectral sequences of double complexes, we obtain the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \mathrm{Tor}_4^S(S/I_4, S/I_1) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & & & \downarrow & & \\
 & & H_4(X.\otimes(S/I_1)) = \mathbb{Z}^s & & & & H^5(Y.\otimes(S/I_1)) = \mathbb{Z}^t & &
\end{array}$$

where all sequences are exact. Since B is a free $\mathbb{Z}$-module, $\mathrm{Tor}_4^S(S/I_4, S/I_1) = A \oplus B$. By Lemma V.3.4, $\mathrm{Tor}_4^S(S/I_4, S/I_1)$ must coincide with $\mathbb{Z}^{s+t}$. Q.E.D.

Consequently, $\beta_4(p, 4)$ is proved to be independent of p. Therefore we have completed the proof of the next theorem.

THEOREM V.3.6. *For any i and n, $\beta_i(p, n)$ does not depend on p.*

So, over an arbitrary commutative ring, there exist the minimal free resolutions of the ideals defined by n-minors of the generic $(n+2)$-square matrices.

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